Matrix and graph orders derived from locally constrained graph homomorphisms

Jiří Fiala[1], Daniël Paulusma[2], and Jan Arne Telle[3]

[1] Charles University, Faculty of Mathematics and Physics,
DIMATIA and Institute for Theoretical Computer Science (ITI) [†],
Malostranské nám. 2/25, 118 00, Prague, Czech Republic.
fiala@kam.mff.cuni.cz

[2] Department of Computer Science, University of Durham,
Science Laboratories, South Road,
Durham DH1 3EY, England.
daniel.paulusma@durham.ac.uk

[3] Department of Informatics, University of Bergen,
N-5020 Bergen, Norway
telle@ii.uib.no

Abstract. We consider three types of locally constrained graph homomorphisms: bijective, injective and surjective. We show that the three orders imposed on graphs by existence of these three types of homomorphisms are partial orders. We extend the well-known connection between degree refinement matrices of graphs and locally bijective graph homomorphisms to locally injective and locally surjective homomorphisms by showing that the orders imposed on degree refinement matrices by our locally constrained graph homomorphisms are also partial orders. We provide several equivalent characterizations of degree (refinement) matrices, e.g. in terms of the dimension of the cycle space of a graph related to the matrix. As a consequence we can efficiently check whether a given matrix M is a degree matrix of some graph and also compute the size of a smallest graph for which it is a degree matrix in polynomial time.

1 Introduction

By graph homomorphisms we mean edge-preserving mappings, i.e. vertex mappings where images of two adjacent vertices are also adjacent in the target graph. Relating pairs of graphs by the existence of a graph homomorphism defines a quasi-order on the class of all graphs, which can be further factorized into a partial order. For a comprehensive survey of these structures see the recent monograph [15].

In this paper we study similar structural properties derived from *locally constrained* graph homomorphisms [10], where for any vertex u the mapping f induces a function from the neighborhood of u to the neighborhood of $f(u)$ which

[†] Supported by the Ministry of Education of the Czech Republic as project 1M0021620808.

is required to be either *bijective* [1,17], *injective* [9,10], or *surjective* [18,13]. See [18,8] for a more general model of locally constrained conditions.

Locally bijective homomorphisms (also known as local isomorphisms or full covers) have important applications, for example in distributed computing [5], in recognizing graphs by networks of processors [2,3], or in constructing highly transitive regular graphs [4]. Locally injective homomorphisms (local epimorphisms or partial covers) are used in distance constrained labelings of graphs [11] and as indicators of the existence of homomorphisms of derivate graphs (line graphs) [21]. Locally surjective homomorphisms (role assignments) are of interest in social network theory where individuals of the same social role relate to other individuals in the same way [7].

Just as in a graph isomorphism, a locally bijective homomorphism maintains vertex degrees and degrees of neighbors and degrees of neighbors of neighbors and so on. The existence of such a mapping from G to H therefore implies equality of the so-called degree refinement matrices of G and H. Since these are easy to compute, they provide both an important necessary condition and a heuristic for the graph isomorphism problem (cf. [19]).

Our results

Degree refinement matrices belong to the class of degree matrices corresponding to degree partitions of the vertex set of a graph. In Sect. 3 we present four equivalent characterizations of degree matrices, e.g. by conditions on the dimension of the cycle space of some matrix-related graph. Given the rather long history and fame of the graph isomorphism problem it is surprising that no characterization of degree (refinement) matrices had been shown previously. As a consequence we can efficiently check whether a given matrix M is a degree matrix or not. We also prove that the size of a smallest graph corresponding to some degree matrix can be computed in polynomial time. In Sect. 4 we prove that the problem whether a given (degree) matrix M is a degree refinement matrix can be solved in polynomial time. In Sect. 5 we introduce three orderings, in which a graph H is smaller than a graph G if a homomorphism from G to H exists, locally constrained to be respectively bijective, injective or surjective. We prove that these are partial orderings and in Sect. 6 we show that these partial orders can be further extended to degree matrices of graphs. These results generalize the use of degree refinement matrices to locally injective and locally surjective homomorphisms. We emphasize that such a relationship was not originally expected, since such degree conditions are not obvious for the non-bijective local constraints.

2 Preliminaries

If not stated otherwise graphs considered in this paper are finite and *simple*, i.e. without loops and multiple edges. For graph terminology not defined below we refer to [6].

For a function $f : V_G \to V_H$ and a set $S \subseteq V_G$ we use the shorthand notation $f(S)$ to denote the image set of S under f, i.e., $f(S) = \{f(u) \mid u \in S\}$. For any $x \in V_H$, the set $f^{-1}(x)$ is equal to $\{u \in V_G \mid f(u) = x\}$.

For a vertex $u \in V_G$ we denote its *neighborhood* by $N_G(u) = \{v \mid (u, v) \in E_G\}$. A k*-regular* graph is a graph, where all vertices have k neighbors (i.e. are of *degree* k). A (k, l)*-regular bipartite* graph is a bipartite graph where vertices of one class of the bi-partition are of degree k and all others are of degree l.

A *graph homomorphism* from $G = (V_G, E_G)$ to $H = (V_H, E_H)$ is a vertex mapping $f : V_G \to V_H$ satisfying the property that for any edge (u, v) in E_G, we have $(f(u), f(v))$ in E_H as well, i.e., $f(N_G(u)) \subseteq N_H(f(u))$ for all $u \in V_G$. Two graphs G and G' are called *isomorphic*, denoted by $G \simeq G'$, if there exists a one-to-one mapping $f : V_G \to V_{G'}$, where both f and f^{-1} are homomorphisms.

Definition 1. *For graphs G and H we denote:*

- $G \xrightarrow{B} H$ *if there exists a so-called* locally bijective homomorphism $f : V_G \to V_H$ *satisfying:*

$$\text{for all } u \in V_G : f(N_G(u)) = N_H(f(u)) \text{ and } |f(N_G(u))| = |N_G(u)|.$$

- $G \xrightarrow{I} H$ *if there exists a so-called* locally injective homomorphism $f : V_G \to V_H$ *satisfying:*

$$\text{for all } u \in V_G : |f(N_G(u))| = |N_G(u)|.$$

- $G \xrightarrow{S} H$ *if there exists a so-called* locally surjective homomorphism $f : V_G \to V_H$ *satisfying:*

$$\text{for all } u \in V_G : f(N_G(u)) = N_H(f(u)).$$

Note that a locally bijective homomorphism is both locally injective and surjective. Hence, any result valid for locally injective or for locally surjective homomorphisms is also valid for locally bijective homomorphisms. We provide an alternative definition of these three kinds of mappings via subgraphs induced by preimages of edges. As far as we know this quite natural definition has not previously appeared in the literature.

Observation 1 *Let $f : G \to H$ be a graph homomorphism. For every edge (u, v) of H, the subgraph of G induced by $f^{-1}(u) \cup f^{-1}(v)$ is a*

- perfect matching *if and only if f is locally bijective,*
- matching *if and only if f is locally injective,*
- bipartite graph without isolated vertices *if and only if f is locally surjective.*

Note that for locally bijective homomorphisms the preimage classes all have the same size and for locally surjective homomorphisms all the preimage classes have size at least one. This yields the following observation:

Observation 2 *If $G \xrightarrow{S} H$, then either $|V_G| > |V_H|$ or else $G \simeq H$.*

For a connected graph G the *universal cover* is defined in [2] as the only (possibly infinite) tree T_G that allows a locally bijective homomorphism $T_G \xrightarrow{B} G$. The vertices of T_G can be represented as walks in G starting in a fixed vertex u that do not traverse the same edge in two consecutive steps. Edges in T_G connect those walks that differ in the presence of the last edge. The mapping $f_0 : T_G \xrightarrow{B} G$ sending a walk in V_{T_G} to its last vertex is a locally bijective homomorphism.

Proposition 1 ([12]). *Let G and H be two connected graphs. From any function $f : G \xrightarrow{I} H$ a locally injective homomorphism $f' : T_G \to T_H$ can be derived. ¿From any function $g : G \xrightarrow{S} H$ a locally surjective homomorphism $g' : T_G \to T_H$ can be derived.*

In the sequel we consider all isomorphism classes of connected simple graphs. We assume that each of these classes is represented by one of its elements, and these representatives form the set $\mathcal{C}$, called the *set of connected graphs.*

3 Degree matrices

Any locally bijective graph homomorphism, with graph isomorphism as a special case, preserves not only vertex degrees but also degrees of neighbors and degrees of neighbors of these neighbors and so on. To capture this property the following notions have been defined (cf. [20, 19]).

Definition 2. *A* degree partition *of a graph G is a partition of the vertex set V_G into blocks $\mathcal{B} = \{B_1, \ldots, B_k\}$ such that whenever two vertices u and v belong to the same block B_i, then for any $j \in \{1, \ldots, k\}$ we have $|N_G(u) \cap B_j| = |N_G(v) \cap B_j| = m_{i,j}$. The $k \times k$ matrix M such that $(M)_{i,j} = m_{i,j}$ is a* degree matrix.

Observe that a graph G can allow several degree matrices, with an adjacency matrix itself being the largest one. *Degree refinement matrices*, which will be considered in the next section, are on the other extreme.

Observation 3 *The vertex set V_G of any graph G that has a $k \times k$ matrix M as one of its degree matrices can be partitioned into $B_1 \cup \ldots \cup B_k$ such that $m_{i,j}|B_i| = m_{j,i}|B_j|$ holds for all $1 \leq i < j \leq k$.*

This immediately implies that for any degree matrix M of size k,

$$m_{i,j} > 0 \text{ if and only if } m_{j,i} > 0 \text{ for all } 1 \leq i < j \leq k.$$

We call integer matrices that have the above property *well-defined.* It is easy to see that there exist well-defined matrices that are not degree matrices of a finite graph. This makes the following decision problem interesting.

Degree Matrix Determination (DMD)
Instance: A square matrix M.
Question: Is M a degree matrix of a finite graph G?

To determine the complexity of DMD we introduce the following definitions. A directed graph $D = (V_D, E_D)$ is called *symmetric* if there exists an arc $(j, i) \in E_D$ whenever there exists an arc $(i, j) \in E_D$. Let $w : E_D \to \mathbb{N}$ be a positive weight function defined on the arc set of D. We call such a graph with positive arc weights a *symmetric directed product graph (sdp-graph)*. We say that a cycle $v_0, v_1, \ldots, v_c, v_0$ in an sdp-graph D has the *cycle product identity* if

$$1 = \prod_{i=0}^{c} \frac{w(v_i, v_{i+1})}{w(v_{i+1}, v_i)},$$

where the subscript of v_{i+1} is computed modulo $c+1$. In other words, a cycle has the cycle product identity if the product of arc weights going clockwise around the cycle is the same as the product counter-clockwise. We say that *the sdp-graph D has the cycle product identity* if every cycle of D has the cycle product identity. Using induction on the cycle length immediately yields:

Observation 4 *An sdp-graph D has the cycle product identity if and only if every induced cycle of D has the cycle product identity.*

For a square matrix M we define the weighted directed graph F_M as follows. Its vertex set V_{F_M} consists of vertices $\{1, \ldots, k\}$. There is an arc from i to $j \neq i$ with weight $m_{i,j}$ if and only if $m_{i,j} \geq 1$. Note that F_M is an sdp-graph if and only if M is well-defined.

Let F'_M be the underlying undirected graph of F_M, i.e., $V_{F'_M} = V_{F_M} = \{1, \ldots, k\}$ and (i, j) is an undirected edge of F'_M, whenever both (i, j) and (j, i) are directed arcs of F_M. We define the *weighted incidence matrix IM* to be the $|E_{F'_M}| \times k$ matrix whose rows are indexed by edges $e = (i, j) \in E_{F'_M}$, $i < j$ and its only non-zero entries in the e-th row are $(IM)_{e,i} = m_{i,j}$ and $(IM)_{e,j} = -m_{j,i}$.

The kernel and rank of a matrix M are denoted by $\ker(M)$ and $\mathrm{rank}(M)$ respectively. The transpose of a matrix M is denoted by M^T. We represent each $e \in E_G$ by a unit vector in the vector space $\mathbb{R}^{|E_G|}$, called the *edge space* $\mathcal{E}_G$ of a graph G. The *cycle space* $\mathcal{S}_G$ of G is the linear subspace of $\mathcal{E}_G$ generated by all cycles in G. We denote the dimension of a linear subspace $\mathcal{D}$ by $\dim(\mathcal{D})$. For every edge e not in a spanning tree T of G there is a unique cycle C_e in the graph $T + e$. Since there are $|E_G| - |V_G| + 1$ of these edges, it is clear that $\dim(\mathcal{S}_G) = |E_G| - |V_G| + 1$.

We now present our characterization of degree matrices.

Theorem 1. *The following statements are equivalent:*

(i) *M is a degree matrix of a graph $G \in \mathcal{C}$.*
(ii) *F_M is a connected sdp-graph satisfying the cycle product identity.*
(iii) *M is well-defined and* $\dim(\ker(IM)) = 1$.
(iv) *M is well-defined and* $\dim(\ker(IM^T)) = \dim(\mathcal{S}_{F'_M})$.

Proof. *(i)* $\Rightarrow$ *(ii)* Since M is a degree matrix, M is well-defined. Hence, F_M is an sdp-graph. Obviously, F_M is connected. Let $C = i_0, \ldots, i_c, i_0$ be a cycle in F_M, where vertex v_i corresponds to block B_i. Use Observation 3 for pairs $(i_0, i_1), \ldots, (i_c, i_0)$ to show that C satisfies the cycle product identity.

(ii) $\Rightarrow$ *(iii)* Since F_M is an sdp-graph, M is well-defined. Consider a path P_{1i} in F_M from the vertex 1 to any vertex i corresponding to the i-th row of M. We apply Observation 3 for consecutive pairs on P_{1i}. Combining these equalities yields a rational $b_i > 0$ such that $|B_i| = b_i|B_1|$ for the blocks B_i and B_1 of any possible graph G with degree matrix M. Because F_M satisfies the cycle product identity, taking another path P'_{1i} between vertices 1 and i would lead to exactly the same equality $|B_i| = b_i|B_1|$. Define $b_1 = 1$. Then any solution of $\ker(IM)$ is a multiple of the vector $\mathbf{b} = (b_1, \ldots, b_k)$.

(iii) $\Rightarrow$ *(i)* We first determine the block sizes of a candidate graph G. We do this with respect to the following two facts. (1) For $p \geq 1$ there exists a p-regular graph on n vertices if and only if $n \geq p + 1$ and np is even. (2) There exists a (p, q)-regular bipartite graph with the degree-p side having m vertices and the degree-q side having n vertices if and only if $m \geq q, n \geq p$ and $mp = nq$. We now choose an integer solution $\mathbf{s}$ of $\ker(IM)$ such that

- $s_i \geq m_{i,i} + 1$ for all i.
- $s_i m_{i,i}$ is even for all i. $\quad (*)$
- $s_i \geq m_{j,i}$ for all i and all $j \neq i$.

Then the following graph G has M as one of its degree matrices. Its vertex set V_G can be partitioned into blocks $B_1 \cup \cdots \cup B_k$ with $|B_i| = s_i$ for all $1 \leq i \leq k$. Its edge set E_G can be chosen such that:

- The subgraph induced by B_i is $m_{i,i}$-regular for $1 \leq i \leq k$.
- The induced bipartite subgraph between vertices of blocks B_i and B_j is $(m_{i,j}, m_{j,i})$-regular for all $1 \leq i < j \leq k$.

(iii) $\Leftrightarrow$ *(iv)* Note that $\dim(\ker(IM)) = 1$ if and only if $\mathrm{rank}(IM^T) = \mathrm{rank}(IM) = k - 1$ if and only if $\dim(\ker(IM^T)) = |E_{F'_M}| - \mathrm{rank}(IM^T) = |E_{F'_M}| - k + 1 = \dim(\mathcal{S}_{F'_M})$. $\square$

Corollary 1. *The* DMD *problem can be solved in polynomial time.*

Proof. First we check whether the matrix M is well-defined. If it is, we construct the graph F_M. Let $M_1, \ldots, M_p$ be the submatrices of M corresponding to the components of F_M. For each M_i we compute $\ker(IM_i)$ and use Theorem 1. $\square$

In this paper we only consider matrices that are the degree matrix of some finite connected graph. If we allow infinite graphs, then we only have to check whether a matrix M is finite and has connected F_M. This is since for any such matrix M we can construct its *universal cover* T_M by taking as root of the (possibly infinite) tree T_M a vertex corresponding to row 1, thus of row-type 1, and inductively adding a new level of vertices while maintaining the property that each vertex of row-type i has exactly $m_{i,j}$ neighbors of row-type j.

Theorem 1 and Corollary 1 immediately imply that for examining whether an sdp-graph has the cycle product identity we do not have to check all (induced) cycles explicitly.

Corollary 2. *The problem whether a symmetric directed graph with positive edge weights has the cycle product identity can be solved in polynomial time.*

Corollary 3. *For any degree matrix M the block sizes of a smallest graph G that has M as one of its degree matrices can be computed in polynomial time.*

Proof. Let $m = \max\{m_{i,j} \mid 1 \leq i, j \leq k\}$. Let $\langle m \rangle$ be the number of bits required to encode m. Then the input size of a $k \times k$ matrix M can be defined as $k^2\langle m \rangle$.

If we compute coefficients b_i as in the proof of Theorem 1, then we find that both nominator and denominator of each b_i have size at most $k\langle m \rangle$. Let α be the product of all denominators of elements b_i. Let b' be a solution of $\ker(IM)$ with entries $b'_i = \alpha b_i$ for all $1 \leq i \leq k$. We divide each b'_i by the greatest common divisor of $b'_1, \dots, b'_k$. This way we have obtained the smallest integer solution $\mathbf{b}^*$ of $\ker(IM)$ in polynomial time. Now we choose the integer γ such that $\gamma \geq \max_{1 \leq i,j \leq k}\{\frac{m_{i,i}+1}{b^*_i}, \frac{m_{j,i}}{b^*_i}\}$, where γ is required to be even if for some i the product $b^*_i m_{i,i}$ is odd. Then $\mathbf{b} = \gamma \mathbf{b}^*$ satisfies all three conditions $(*)$, i.e., it yields the block sizes of a smallest graph G in the same way as in the proof of Theorem 1. (The size of G itself might be exponential in $\langle \mathbf{b} \rangle$.) $\square$

4 Degree refinement matrices

For many pairs of graphs (G, H) we can easily determine that a locally bijective homomorphism from G to H does not exist.

Definition 3. *The* d*egree refinement matrix* $\mathrm{drm}(G)$ *of* G *is the degree matrix corresponding to the canonical (as explained below) coarsest degree partition of* G*, i.e., with the fewest number of blocks.*

If $\mathrm{drm}(G) \neq \mathrm{drm}(H)$ then no locally bijective homomorphism exists between G and H, and this condition can be checked by computing both minimum degree partitions by procedure MDP Construction that runs in $\mathcal{O}(n^3)$ time (cf. [2]).

MDP Construction
Input: A graph G.
Output: The minimal degree partition $\mathcal{B}$.

0. Set $\mathcal{B}^0 = \{B^0_1\} = \{V_G\}$, $t = 1$.
1. For each vertex u compute the degree vector $\overrightarrow{d(u)} := \Big(|N(u) \cap B^t_1|, |N(u) \cap B^t_2|, \dots\Big)$.
2. Set $t := t + 1$ and define the new partition $\mathcal{B}^t$ of V_G such that
   - $u, v \in B^t_i$ if and only if $\overrightarrow{d(u)} = \overrightarrow{d(v)}$,
   - $u \in B^t_i$, $v \in B^t_{i'}$ with $i < i'$ if and only if
     - either $u \in B^{t-1}_j$, $v \in B^{t-1}_{j'}$ with $j < j'$,
     - or $u, v \in B^{t-1}_j$ and $\overrightarrow{d(u)} >_{\mathrm{Lex}} \overrightarrow{d(v)}$,

   where $>_{\mathrm{Lex}}$ is the lexicographic order on integer sequences.
3. If $\mathcal{B}^t = \mathcal{B}^{t-1}$ then set $\mathcal{B} = \mathcal{B}^t$ and stop, otherwise continue by step 1.

We modify this procedure into the efficient algorithm DRM CONSTRUCTION. Given a degree matrix M it computes a matrix M' such that $M' = \mathrm{drm}(G)$ for any graph G with degree matrix M. Moreover, given a graph G it computes the degree refinement matrix of G when we take an adjacency matrix of G as its input. Note that in steps **2** and **3** the canonical order of the blocks is defined.

DRM CONSTRUCTION
Input: A degree partition matrix M.
Output: A degree refinement matrix M' that encodes all graphs with degree matrix M.

0. Set $\mathcal{R}^0 = \{R_1^0\} = \{1, \ldots, k\}$, $t = 1$.
1. For each row $r = 1, \ldots, k$ compute the row-degree vector $\overrightarrow{d(r)} := \left(\sum_{i \in R_1^t} m_{r,i}, \sum_{i \in R_2^t} m_{r,i}, \ldots\right)$.
2. Set $t := t + 1$ and define the new partition $\mathcal{R}^t$ of $\{1, \ldots, k\}$ such that
- $r, s \in B_i^t$ if and only if $\overrightarrow{d(r)} = \overrightarrow{d(s)}$,
- $r \in B_i^t$, $s \in B_{i'}^t$ with $i < i'$ if and only if
  - either $r \in B_j^{t-1}$, $s \in B_{j'}^{t-1}$ with $j < j'$,
  - or $r, s \in B_j^{t-1}$, and $\overrightarrow{d(r)} >_{\mathrm{Lex}} \overrightarrow{d(s)}$.

3. If $\mathcal{R}^t = \mathcal{R}^{t-1}$ then set $M' = \begin{pmatrix} \overrightarrow{d(r)} : r \in R_1^t \\ \overrightarrow{d(r)} : r \in R_2^t \\ \vdots \end{pmatrix}$ and stop,
otherwise continue by step 1.

By applying the above algorithm and Corollary 1 we immediately obtain the following.

Theorem 2. *Checking if a given matrix M is a degree refinement matrix can be done in polynomial time.*

5 Partial orders on graphs

It is well-known that graph homomorphisms define a quasiorder on the class of all graphs, which can be factorized into a partial order. For an overview of these results see the recent monograph [16]. We show that a similar interesting structure exists on the class of connected graphs $\mathcal{C}$ for locally constrained homomorphisms. For this purpose we will view $\xrightarrow{B}$, $\xrightarrow{I}$ and $\xrightarrow{S}$ as binary relations on $\mathcal{C}$, denoted by $(\mathcal{C}, \xrightarrow{*})$ if necessary, where $*$ will indicate the appropriate local constraint. We show that $(\mathcal{C}, \xrightarrow{*})$ is a partial order for any of the three local constraints $* = B, I, S$.

Observe first that for any $G \in \mathcal{C}$ the identity mapping $i : V_G \to V_G$ clarifies that all three relations $\xrightarrow{*}$ are *reflexive*.

The composition of two graph homomorphisms of the same kind of local constraint (B, I, S) is again a graph homomorphism of the same kind. Hence each $\xrightarrow{*}$ is also *transitive*.

For antisymmetry, suppose for $G, H \in \mathcal{C}$ that $f : G \xrightarrow{*} H$, $g : H \xrightarrow{*} G$, where f, g are of the same local constraint. For $* = B, S$ we can invoke Observation 2 to conclude that $G \simeq H$. For $* = I$ we use the following result.

Theorem 3 ([12]). *Let G be a (possibly infinite) graph and let H be a graph in $\mathcal{C}$. If G allows both a locally injective and a locally surjective homomorphism to H, then both these homomorphisms are locally bijective.*

For $* = I$ we have $g \circ f : G \xrightarrow{I} G$ and $G \xrightarrow{S} G$ by the identity mapping. By Theorem 3 the mapping $g \circ f$ is locally bijective. Since G is connected, $(g \circ f)(V_G) = V_G$ implying that f is (globally) injective. By the same kind of arguments we deduce that g is injective. This means that f is surjective, and hence f is a graph isomorphism from G to H. Hence, all three relations are *antisymmetric*. We would like to mention that the antisymmetry of $\xrightarrow{I}$ also follows from an iterative argument of [21].

Combining the results above with Theorem 3 yields the following.

Theorem 4. *All three relations $(\mathcal{C}, \xrightarrow{B}), (\mathcal{C}, \xrightarrow{I})$ and $(\mathcal{C}, \xrightarrow{S})$ are partial orders with $(\mathcal{C}, \xrightarrow{B}) = (\mathcal{C}, \xrightarrow{I}) \cap (\mathcal{C}, \xrightarrow{S})$.*

6 Partial orders on degree refinement matrices

We again recall the fact that a locally bijective homomorphism from a graph G to a graph H may exist only if G and H have the same degree refinement matrix.

Theorem 5 ([19]). *Two graphs $G, H \in \mathcal{C}$ have a common degree refinement matrix if and only if their universal covers are isomorphic as well as if and only if there exists a graph $F \in \mathcal{C}$ allowing locally bijective homomorphisms to both G and H.*

In view of this theorem we can also define the universal cover T_M associated with a degree refinement matrix M as the universal cover $T_G = T_M$ of any graph G with $\mathrm{drm}(G) = M$. This implies that the symmetric and transitive closure of the partial order $(\mathcal{C}, \xrightarrow{B})$ is an equivalence relation whose classes can be naturally represented by degree refinement matrices. It is natural to ask if the other two kinds of locally constrained homomorphisms are also conditioned by the existence of a well-defined relation on the degree refinement matrices. Here, we prove that such a relation exists and moreover, that it is a partial order.

Definition 4. *We denote the set of all degree refinement matrices of graphs in $\mathcal{C}$ by $\mathcal{M}$. We define three relations $\xrightarrow{B}$, $\xrightarrow{I}$, and $\xrightarrow{S}$ respectively, on $\mathcal{M}$ as follows. For two matrices $M, N \in \mathcal{M}$ we have $M \xrightarrow{*} N$ if there exist graphs $G \in \mathcal{C}$ with $\mathrm{drm}(G) = M$ and $H \in \mathcal{C}$ with $\mathrm{drm}(H) = N$ such that $G \xrightarrow{*} H$ holds for the appropriate local constraint.*

As stated above $(\mathcal{M}, \xrightarrow{B})$ is a trivial order where no two distinct elements are comparable. For the other two relations, the *reflexivity* of the relation follows directly from the existence of the identity mapping on any underlying graph. Antisymmetry and transitivity require more effort.

For proving *antisymmetry* we involve the notion of universal cover. Assume that $M \xrightarrow{I} N$ and $N \xrightarrow{I} M$. By Proposition 1, there exist locally injective homomorphisms $f' : T_M \to T_N$ and $g' : T_N \to T_M$. Recall from Sect. 2 that there exist a locally bijective homomorphism $f_0 : T_M \to G_1$. As in the previous section we now invoke Theorem 3 to conclude that $f_0 \circ g' \circ f' : T_M \xrightarrow{I} G_1$ is locally bijective. This implies that both f' and g' are locally bijective, and consequently the universal covers T_M and T_N are isomorphic. Hence $M = N$ due to Theorem 5. The antisymmetry of $\xrightarrow{S}$ can be proven according to exactly the same arguments.

The transitivity property of $\xrightarrow{I}$ follows directly from the next lemma.

Lemma 1. *Let $G_1, G_2, H_1, H_2 \in \mathcal{C}$ be such that $G_1 \xrightarrow{I} H_1$ and $G_2 \xrightarrow{I} H_2$, where H_1 and G_2 share the same degree refinement matrix. Then there exists a graph $F \in \mathcal{C}$ such that $F \xrightarrow{I} H_2$ and $F \xrightarrow{B} G_1$.*

Proof. Using Theorem 5 we first construct a finite graph F' such that $F' \xrightarrow{B} H_1$ and $F' \xrightarrow{B} G_2$. The projection $\pi_2 : F' \xrightarrow{B} G_2$ composed with the locally injective homomorphism $g : G_2 \xrightarrow{I} H_2$ gives that $F' \xrightarrow{I} H_2$. See Fig. 1.

$$\begin{tikzcd} F \arrow[r, "f'"] \arrow[d, "\pi"'] & F' \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ G_1 \arrow[r, "f"] & H_1 & G_2 \arrow[r, "g"] & H_2 \end{tikzcd}$$

$$\text{drm}: \quad M \qquad N \quad \dots \quad N \qquad Q$$

Fig. 1. Commutative diagram for transitivity of $\xrightarrow{I}$ where horizontal mappings are injective and others are bijective.

As $F' \xrightarrow{B} H_1$ via projection π_1, by Observation 1 the preimage $\pi_1^{-1}(x)$ has the same size for all vertices $x \in V_{H_1}$, say k. We assume that all vertices of F' that map onto a vertex x are labeled $\{x_1, x_2, \ldots, x_k\}$.

The vertex set of the desired graph F is the Cartesian product $V_{G_1} \times \{1, \ldots, k\}$. For simplicity we abbreviate (u, i) as u_i. Define the edges of F as follows:

$$(u_i, v_j) \in E_F \Leftrightarrow (u, v) \in E_{G_1} \text{ and } (f(u)_i, f(v)_j) \in E_{F'}.$$

We define two mappings $f' : u_i \to f(u)_i$ and $\pi : u_i \to u$. According to Observation 1, f' is a locally injective homomorphism from F to F' and π is a locally bijective homomorphism from F to G_1. The mapping $g \circ \pi_2 \circ f'$ is a locally injective homomorphism $F \xrightarrow{I} H_2$. □

The same assertion can be proven for the order $\xrightarrow{S}$ with exactly the same arguments, the only difference is that the preimage in F of any edge $(x_i, y_j) \in E_{F'}$ is a spanning bipartite graph.

Theorem 6. *For any constraint $* = B, I, S$ the relation $(\mathcal{M}, \xrightarrow{*})$ is a partial order. It arises as a factor of the order $(\mathcal{C}, \xrightarrow{*})$, when we unify the graphs that have the same degree refinement matrices.*

Any locally injective homomorphism $G \xrightarrow{I} H$ can be extended to a locally bijective homomorphism $G' \xrightarrow{B} H$, where $G \subseteq G'$ [17]. This yields an alternative definition of the order $(\mathcal{M}, \xrightarrow{I})$: For matrices M, N holds $M \xrightarrow{I} N$ if and only if there exists graphs G and H with degree refinement matrices M and N, respectively, such that G is a subgraph of H. This straightforwardly implies the first claim of the observation below. The second claim (and the first claim as well) follows by Proposition 1 and a simple inductive argument on the two trees T_M and T_N.

Observation 5 *For any degree refinement matrices $M, N \in \mathcal{M}$ it holds that if $M \xrightarrow{I} N$ then $T_M \subseteq T_N$, and if $M \xrightarrow{S} N$ then $T_N \subseteq T_M$.*

The reverse is not true: for $\xrightarrow{S}$ take $M = \mathrm{drm}(P_4)$ and $N = \mathrm{drm}(P_3)$. The counterexample for $\xrightarrow{I}$ requires a bit more effort (see [14]).

Theorem 3 can now be translated to matrices. If $M \xrightarrow{I} N$ and $M \xrightarrow{S} N$, then $M \xrightarrow{B} N$, i.e., $M = N$.

Corollary 4. $(\mathcal{M}, \xrightarrow{B}) = (\mathcal{M}, \xrightarrow{I}) \cap (\mathcal{M}, \xrightarrow{S}) = (\mathcal{M}, \{(M, M) : M \in \mathcal{M}\})$.

Proof. Suppose $G_1 \xrightarrow{I} H_1$ and $G_2 \xrightarrow{S} H_2$ hold with $\mathrm{drm}(G_i) = M$ and $\mathrm{drm}(H_i) = N$ $(i = 1, 2)$. By Observation 5, we have that $T_M \subseteq T_N$ and $T_N \subseteq T_M$. We represent these inclusions by locally injective homomorphisms $f' : T_M \to T_N$ and $g' : T_N \to T_M$. Then we may conclude $M = N$ by the same arguments as in the proof of antisymmetry of $\xrightarrow{I}$. □

7 Conclusion

We have proved that graph homomorphisms with local constrains between finite graphs impose interesting orders on the class of degree matrices. We have also shown that such matrices can be easily detected and, moreover, a canonical representative of a class of equivalent matrices can be computed by an efficient algorithm. The generalization of these concepts beyond the class of degree matrices of finite graphs and their applications in theoretical computer science is subject of further study.

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