Algorithms for comparability of matrices in partial orders imposed by graph homomorphisms

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Abstract. Degree refinement matrices have tight connections to graph homomorphisms that locally, on the neighborhoods of a vertex and its image, are constrained to three types: bijective, injective or surjective. If graph G has a homomorphism of given type to graph H, then we say that the degree refinement matrix of G is smaller than that of H. This way we obtain three partial orders. We present algorithms that will determine whether two matrices are comparable in these orders. For the bijective constraint no two distinct matrices are comparable. For the injective constraint we give a PSPACE algorithm, which we also apply to disprove a conjecture on the equivalence between the matrix orders and universal cover inclusion. For the surjective constraint we obtain some partial complexity results.

1 Introduction

Graph homomorphisms, originally obtained as a generalization of graph coloring, have a great deal of applications in computer science and other fields. Beyond these computational aspects they impose an interesting structure on the class of graphs, with many important categorical properties, see e.g. the recent monograph [6]. We focus our attention on graph homomorphisms with local constraints. Originally arising in topological graph theory, these homomorphisms were required to act as a bijection on the neighborhood of each vertex [2]. We consider further local constraints, namely local injectivity or local surjectivity. Both these kinds of homomorphisms have already been studied due to their applications in models of telecommunication [4] and in social science [3, 8].

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In related work [5] we have shown that these locally constrained homomorphisms impose an algebraic structure on the class of connected finite graphs. We also extended a necessary condition for the existence of a locally bijective homomorphism between two graphs [7] to a similar but much more sophisticated statement for locally injective or surjective homomorphisms. An important role in this characterization was predicated to matrices that describe the degree structure of a graph, the so-called degree refinement matrices. We gave a characterization of these matrices, and showed that both locally injective and locally surjective graph homomorphisms impose partial orders on degree refinement matrices [5].

New Results

In this paper we continue this work and turn our attention away from categorical questions to focus instead on the following computational questions: Given two degree matrices, are they comparable in the partial orders imposed by local injectivity or surjectivity? It is not obvious that these questions are decidable, and indeed for local surjectivity we must leave this as a major open problem. However, for local injectivity we manage to show an upper bound on the size of the smallest graphs that can possibly justify a positive answer and use this to provide a PSPACE algorithm. The existence of a locally bijective homomorphism between two graphs is conditioned by the equivalence of their degree refinement matrices, which can also be expressed as an isomorphism between their universal covers [7]. For the other two kinds of locally constrained homomorphisms this naturally raises the question, and conjecture, of a similar tight relationship between matrix comparison in the partial order and inclusion of universal covers. However, we apply our PSPACE algorithm to disprove this enticing conjecture. For the surjective constraint we obtain some partial results on the complexity of matrix comparison.

2 Preliminaries

Graphs considered in this paper are simple, i.e. with no loops and multiple edges, connected and, if not stated otherwise, they are also finite. We denote the class of such graphs by $\mathcal{C}$. For any vertex $u \in V_G$ the symbol $N(u)$ denotes the *neighborhood* of u, i.e. the set of all vertices adjacent to u. A k-regular graph is a graph, where all vertices have k neighbors (i.e. are of *degree* k). A (k, l)-regular bipartite graph is a bipartite graph where vertices of one class of the bi-partition are of degree k and the remaining vertices are of degree l. A graph G is a *subgraph* of a graph H if $V_G \subseteq V_H$ and $E_G \subseteq E_H$. This is denoted by $G \subseteq H$.

A *degree partition* of a graph G is a partition of the vertex set V_G into blocks $\mathcal{B} = \{B_1, \dots, B_k\}$ such that whenever two vertices u and v belong to the same block B_i, then for any $j \in \{1, \dots, k\}$ we have $|N_G(u) \cap B_j| = |N_G(v) \cap B_j| = m_{i,j}$. The $k \times k$ matrix M such that $(M)_{i,j} = m_{i,j}$ is a *degree matrix*. A graph G can allow several degree matrices. The matrix that corresponds to the partition with the *smallest* number of blocks and where these blocks follow the so-called canonical ordering (just some ordering to provide uniqueness) is called its *degree refinement matrix*. It is denoted by $\mathrm{drm}(G)$ for a graph G and computed in polynomial time by a simple stepwise refinement starting from an initial partition by vertex degrees with blocks ordered by increasing degrees. The refinement of the partition continues until any two nodes in the same block have the same number of neighbors in any other block, see e.g. [5]. (See Fig. 1 for an example.) We denote the class of degree refinement matrices of graphs in $\mathcal{C}$ by $\mathcal{M}$.

A graph homomorphism is an edge-preserving mapping $f : V_G \to V_H$, i.e. $(f(u), f(v))$ is an edge of H whenever $(u, v) \in E_G$. A homomorphism $f : G \to H$ may be further confined to adhere to some local constraints, as in the following definition.

Definition 1. *We call a graph homomorphism $f : G \to H$* locally bijective, locally injective *or* locally surjective *if for every vertex $u \in V_G$ the restriction of f to $N(u)$ is a bijection, injection or surjection between $N(u)$ and $N(f(u))$, respectively. We denote it as $f : G \xrightarrow{B} H$ or $f : G \xrightarrow{I} H$ or $f : G \xrightarrow{S} H$, respectively.*

For each of the three types of local constraints $* = B$ (bijective), $* = I$ (injective) or $* = S$ (surjective), we will in this paper focus on the following three relations on the class of degree refinement matrices $\mathcal{M}$:

$$M \xrightarrow{*} N \iff \text{exist } G, H \in \mathcal{C} : \mathrm{drm}(G) = M,\ \mathrm{drm}(H) = N \text{ and } G \xrightarrow{*} H$$

In [5] we showed that all three relations $(\mathcal{M}, \xrightarrow{B})$, $(\mathcal{M}, \xrightarrow{I})$ and $(\mathcal{M}, \xrightarrow{S})$ are partial orders. Note that $(\mathcal{M}, \xrightarrow{B})$ is in fact a trivial order, since in [7] it has been shown that $drm(G) = drm(H)$ is a necessary condition for $G \xrightarrow{B} H$.

For a graph $G \in \mathcal{C}$ the *universal cover* T_G is defined in [1] as the only (possibly infinite) tree that allows $T_G \xrightarrow{B} G$. The vertices of T_G can be represented as walks in G starting in a fixed vertex u that do not traverse the same edge in two consecutive steps. Edges in T_G connect those walks that differ in the presence of the last edge. The mapping $T_G \xrightarrow{B} G$ sending a walk in V_{T_G} to its last vertex is a locally bijective homomorphism. Universal covers are in one-to-one correspondence with degree refinement matrices, hence for $M \in \mathcal{M}$ we can define $T_M = T_G$ for any G with $\mathrm{drm}(G) = M$.

Proposition 1 ([5]). *The relation $M \xrightarrow{I} N$ holds if and only if there exist graphs G with $\mathrm{drm}(G) = M$ and H with $\mathrm{drm}(H) = N$ such that $G \subseteq H$.*

We use the following relationship between degree refinement matrices and universal covers.

Proposition 2 ([5]). *For any degree refinement matrices $M, N \in \mathcal{M}$ it holds that if $M \xrightarrow{I} N$ then $T_M \subseteq T_N$, and if $M \xrightarrow{S} N$ then $T_N \subseteq T_M$.*

For computational complexity purposes $\langle X \rangle$ denotes the size of the instance X (graph, matrix, etc.) in usual binary encoding of numbers. Formally we represent vertices of a graph G by numbers $\{1, 2, \ldots, |V_G|\}$ and its edges as a list of its vertices. A graph with m edges on n vertices hence requires space $\langle G \rangle = \Theta(m \log n)$.

For an integral-valued $k \times l$ matrix A let $a^* = 2 + \max\{|A_{i,j}| \mid 1 \leq i \leq k \text{ and } 1 \leq j \leq l\}$. Then the size of A is given by $\langle A \rangle = \Theta(kl \log a^*)$.

We will need the following technical lemma for our PSPACE algorithm.

Lemma 1. *Let A be an integral-valued $k \times l$ matrix with $l > k$. If $A\mathbf{x} = \mathbf{0}$ allows a nontrivial nonnegative solution, then it allows a nontrivial nonnegative integer solution $\mathbf{x}$ with at most $k+1$ nonzero entries and with $\langle x_i \rangle = O(k \log(ka^*))$ for each entry x_i.*

Proof. If a solution $\mathbf{x}$ with more than $k+1$ positive coefficients exists, then the columns corresponding to $k+1$ of these variables are linearly dependent. Let the coefficients of such a linear combination form a vector $\mathbf{x}'$. Obviously $A\mathbf{x}' = \mathbf{0}$, but the entries of $\mathbf{x}'$ may not be necessarily nonnegative.

Without loss of generality we assume that at least one of the entries in $\mathbf{x}'$ is positive. Then, for a suitable value $\alpha = -\min\{\frac{x_i}{x'_i} \mid x'_i > 0\}$ the vector $\mathbf{x} + \alpha\mathbf{x}'$ is also a nontrivial nonnegative solution with more zero entries than $\mathbf{x}$.

Repeating this trimming iteratively we obtain a nontrivial nonnegative solution with at most $k+1$ nonzero entries. As the other entries are zero, we may restrict the matrix A to columns corresponding to nonzero entries of the solution. It may happen that the rank of the modified matrix decreases. Then we reduce the number of rows until the remaining ones become linearly independent. By repeating the above process we finally get an $k' \times (k'+1)$ matrix B of rank $k' \leq k$, such that $B\mathbf{y} = \mathbf{0}$ allows a nontrivial nonnegative solution $\mathbf{y}$. Such $\mathbf{y}$ can be extended to a solution $\mathbf{x}$ of the original system by inserting zero entries.

Without loss of generality we assume that the first k' columns of B are linearly independent, and we arrange them in a regular matrix R. Then its inverse can be expressed as $R^{-1} = \frac{adj(R)}{\det(R)}$, where $adj(R)$ is the adjoint matrix of R. By the determinant expansion we have that $\det(R) \leq k'!(a^*)^{k'} \leq k!(a^*)^k \leq k^k(a^*)^k$. Then we find that $\langle \det(R) \rangle = O(k \log(ka^*))$. Each element of $adj(R)$ is a determinant of a minor of R and hence is smaller than $(k-1)^{k-1}(a^*)^{k-1}$.

Now consider the integral valued matrix $B' = \det(R) \cdot R^{-1}B$. Then

- $\mathbf{y}$ is a solution of $B'\mathbf{y} = \mathbf{0}$ if and only if $B\mathbf{y} = \mathbf{0}$.
- The first k' columns of B' form the matrix $\det(R) \cdot I_{k'}$.
- In the last column the entries $z_1, \ldots, z_l$, are all negative (if $\det(R) > 0$) or all positive (otherwise).

If $\det(R) > 0$ then $\mathbf{y} = (-z_1, \ldots, -z_{k'}, \det(R))$ is a nonnegative nontrivial integral solution to $B\mathbf{y} = \mathbf{0}$. In the other case we swap the sign and choose $\mathbf{y} = (z_1, \ldots, z_{k'}, -\det(R))$. As each $z_i \leq ka^* \max_{ij}(adj(R)_{ij}) \leq k^k(a^*)^k$, we obtain $\langle z_i \rangle = O(k \log(ka^*))$, which concludes the proof. $\square$

3 Matrix comparison via local injectivity

In this section we consider the problem of deciding whether for given degree refinement matrices M and N the comparison $M \xrightarrow{I} N$ holds.

Observe that according to the definition of the order $(\mathcal{M}, \xrightarrow{I})$, there is no obvious bound on the sizes of graphs G and H with M and N as degree refinement matrices that should justify the comparison $M \xrightarrow{I} N$.

The main result of this paper is the following theorem:

Theorem 1. *Let M, N be degree refinement matrices of order k and l. If $M \xrightarrow{I} N$, then there exist a graph G of size $(klm^*)^{O(k^2l^2)}$ and a graph H of size $(klm^*n^*)^{O(k^2l^2)}$ such that $G \xrightarrow{I} H$, $\mathrm{drm}(G) = M$ and $\mathrm{drm}(H) = N$.*

Proof. Throughout this proof we assume that indices i, j, r, s used later always belong to feasible intervals $1 \le i, r \le k$ and $1 \le j, s \le l$. For clarity we often abbreviate pairs of sub-/super-scripts i, j by ij, so in this notation, ij does not mean multiplication.

The main idea of the construction is as follows. Assume that $M \xrightarrow{I} N$ holds. Then by Proposition 1 there exist a graph H and a subgraph $G \subseteq H$ witnessing $M \xrightarrow{I} N$. Let $\{U_1, \dots, U_k\}$ be the degree partition of G and $\{V_1, \dots, V_l\}$ the one for H. We further partition $V_G \subseteq V_H$ as follows. For each pair of indices r and s we define the set

$$W_{rs} = \{v \mid v \in U_r \cap V_s\},$$

and for some vertex $w \in W_{rs}$ we can write a vector describing the distribution of neighbors of w in the classes $W_{11}, \dots, W_{kl}$.

We first show that for given M and N the set T containing all such vectors is finite. Then, with help of T, we design a set of equations that allows a solution if and only if the desired graphs G and H exist. As the size of T is bounded, we can establish the desired bounds on the size of G and H.

Let $\mathbf{p}^{rs}$ be a vector of length kl whose entries are positive integers and are indexed by pairs ij. If the vector $\mathbf{p}^{rs}$ further satisfies

$$\sum_{j=1}^{l} p_{ij}^{rs} = m_{ri} \qquad \text{for all } 1 \le i \le k, \tag{1}$$

$$\sum_{i=1}^{k} p_{ij}^{rs} \le n_{sj} \qquad \text{for all } 1 \le j \le l, \tag{2}$$

then we call $\mathbf{p}^{rs}$ an *injective distribution row for indices r and s*. Note that for given matrices M and N and any feasible choice of r, s the number of all different injective distribution rows for r and s is finite. We denote the set of all injective distribution rows for indices r and s by

$$T(r, s) = \{\mathbf{p}^{rs(1)}, \dots, \mathbf{p}^{rs(t(rs))}\}.$$

Due to (1), the number of distribution rows for every $\mathbf{p}^{rs}$ is bounded by $t(r, s) \le \binom{m^*+l-1}{m^*}^k = O((m^*+1)^{kl})$. The total number of distribution rows is then

$$t_0 = \sum_{r,s} t(r, s) = O(kl(m^*+1)^{kl}).$$

Now consider a set of t_0 variables $w^{rs(t)}$ for all feasible r, s and all $1 \leq t \leq t(r,s)$. We claim that the existence of a nontrivial *nonnegative* solution of the following homogeneous system of $k^2 l^2$ equations in t_0 variables:

$$\sum_{t=1}^{t(r,s)} p_{ij}^{rs(t)} w^{rs(t)} = \sum_{t'=1}^{t(i,j)} p_{rs}^{ij(t')} w^{ij(t')} \qquad 1 \leq i, r \leq k,\ 1 \leq j, s \leq l \tag{3}$$

is a necessary and sufficient condition for the existence of finite graphs G and H witnessing $M \xrightarrow{I} N$.

Necessity: For given G and H we assume without loss of generality that $G \subseteq H$. Firstly determine the sets W_{rs}, and for each vertex $u \in W_{rs} \subseteq V_G$ compute the distribution vector of its neighbors $\mathbf{p}(u) = (|N(u) \cap W_{11}|, \ldots, |N(u) \cap W_{kl}|)$. Then the vector $\mathbf{w}$ with entries $w^{rs(t)} = |\{u : \mathbf{p}(u) = \mathbf{p}^{rs(t)}\}|$ is a nontrivial solution of (3), since in each equation both sides are equal to the number of edges connecting sets W_{rs} and W_{ij}.

Sufficiency: Assume that the system (3) has a nontrivial nonnegative solution. By appropriate scaling we obtain a nonnegative integer solution $\mathbf{w} = (w^{11(1)}, \ldots, w^{kl(t(k,l))})$ with each $w^{rr(t)}$ is even.

We first build a multigraph G_0 upon t_0 sets of vertices $W^{11(1)}, \ldots, W^{kl(t(k,l))}$, where $|W^{rs(t)}| = w^{rs(t)}$ (some sets may be empty) as follows: Denote $W^{rs} = W^{rs(1)} \cup \cdots \cup W^{rs(t(r,s))}$.

Our choice of even values $w^{rr(t)}$ allows us to build an arbitrary $p_{rr}^{rr(t)}$-regular multigraph on each set $W^{rr(t)}$.

As $\mathbf{w}$ satisfies (3), we can easily build a bipartite multigraph between any pair of different sets W^{rs} and W^{ij} such that the number of edges between them is equal to $\sum_{t=1}^{t(r,s)} p_{ij}^{rs(t)} w^{rs(t)} = \sum_{t'=1}^{t(i,j)} p_{rs}^{ij(t')} w^{ij(t')}$.

For any vertex u in $W^{rs(t)}$ with more than $p_{ij}^{rs(t)}$ neighbors in W^{ij} there exists a vertex u^* in some $W^{ij(t^*)}$ with less than $p_{ij}^{rs(t^*)}$ neighbors, and vice versa. Now we remove an edge between u and some neighbor $v \in W^{ij}$ and add the edge (u', v). We repeat this procedure until all vertices of W^{rs} have the right number of neighbors in W^{ij}. Then we do the same for vertices in W^{ij}.

This way we have constructed a bipartite multigraph between W^{rs} and W^{ij} such that each vertex of each $W^{rs(t)}$ is incident with exactly $p_{ij}^{rs(t)}$ edges and each vertex of each $W^{ij(t')}$ is incident with exactly $p_{rs}^{ij(t')}$ edges.

It may happen in some instances that multiple edges are unavoidable. In that case let $d \leq m^*$ be the maximal edge multiplicity in G_0. We obtain the graph G by taking d copies of the multigraph G_0 and replace each collection of d parallel edges of multiplicity $d' \leq d$ by a simple d'-regular bipartite graph.

Due to the construction, it is straightforward to check that vertices from sets that share the same index r form the r-th block of the degree partition of G and that $\mathrm{drm}(G) = M$.

For the construction of H we first distribute the vertices of G into sets $V'_1, \dots, V'_l$, where

$$V'_s = \bigcup_{r=1}^{k} \bigcup_{t=1}^{t(r,s)} W^{rs(t)}.$$

Since N is a degree refinement matrix, the following homogeneous system whose equations represent the number of edges between two different blocks in N has nontrivial solutions:

$$n_{sj} v_s = n_{js} v_j \qquad 1 \le j, s \le l \tag{4}$$

Then we form sets $V_1, \dots, V_l$ by further inserting new vertices into $V'_1, \dots, V'_l$ until for each $s, j : |V_s| n_{sj} = |V_j| n_{js}$ and $|V_s| > 0$ is even.

Next we build a multigraph H_0 by constructing an (n_{sj}, n_{js})-regular bipartite multigraph between any two sets V_s and V_j, and an n_{jj}-regular multigraph on each V_j. In case multiple edges cannot be avoided we take sufficient copies of H_0 and make the appropriate reparations. So we perform these steps in the same way as before, however without removing any edges between vertices in (any copy of) G.

Clearly, G is a subgraph of the resulting graph H and H has N as its degree refinement matrix.

To conclude the proof of the theorem we discuss the size of G and H. Note that all coefficients $p_{ij}^{rs(t)}$ of system (3) are at most m^*. Then, by Lemma 1, we find a nontrivial nonnegative integer solution $\mathbf{w}$ whose entry sizes are bounded by $O(k^2 l^2 \log(klm^*))$.

We can use the entries of $2\mathbf{w}^*$ for the sizes of the blocks in the multigraph G_0. Since we take at most m^* copies of G_0 to obtain our final graph G, we find that $\langle G \rangle = (klm^*)^{O(k^2 l^2)}$.

Analogously, the size of each entry of a solution $\mathbf{v}$ of system 4 is bounded by $O(l^2 \log(ln^*))$. Since multigraph H_0 must contain graph G, we use the entries of $\langle G \rangle$ for the block sizes of H_0. We need at most n^* copies of H_0 for graph H. Hence, each block size $|V_i|$ can be chosen within the upper bound $\langle G \rangle \cdot (ln^*)^{O(l^2)}$ implying that $\langle H \rangle = (klm^* n^*)^{O(k^2 l^2)}$. □

We can now settle the first computational complexity result for the following matrix comparison problem:

Matrix Injectivity (MI)
Instance: Degree refinement matrices M and N.
Question: Does $M \xrightarrow{I} N$ hold?

Corollary 1. *The* MI *problem is decidable in polynomial space.*

Proof. The proof of Theorem 1 showed that $M \xrightarrow{I} N$ if and only if system (3) has a nontrivial nonnegative solution. Then by Lemma 1 there exists a nontrivial nonnegative integral solution with at most $k^2 l^2 + 1$ nonzero entries, which are each bounded in size by $O(k^2 l^2 \log(klm^*))$.

So we only have to consider vectors of this form. As the size of any such vector is polynomial, we can by brute force sequentially list them all, and test their feasibility for (3). Note that any restriction of (3) to polynomially many columns can be generated in PSPACE as well. □

As we have discussed in the introduction, the matrix order $(\mathcal{M}, \xrightarrow{I})$ was considered as a nontrivial necessary condition for the decision problem whether $G \xrightarrow{I} H$. As the size of M and N should vary from being independent in the size of the given graphs to be of approximately the same size of G, H, even the exponential time-complexity of the MI problem might be plausible as a precomputation for some instances.

We apply Theorem 1 to disprove the following interesting conjecture on the equivalence between comparison of degree matrices in $\xrightarrow{I}$ and inclusion of universal covers.

Conjecture 1. For any two degree refinement matrices M and N the following equivalence holds: $\quad M \xrightarrow{I} N \quad \Longleftrightarrow \quad T_M \subseteq T_N.$

We note here that the affirmative answer for the only if implication was already shown in Proposition 2. The following example acts both as an example for the application of Theorem1, and as an counterexample of Conjecture 1.

Corollary 2. *There exist matrices M and N such that $T_M \subseteq T_N$, but $M \not\xrightarrow{I} N$.*

Proof. We first construct graphs G and H such that $H \xrightarrow{S} G$. Denote $M = \mathrm{drm}(G)$ and $N = \mathrm{drm}(H)$. Then according to Proposition 2 we get that $T_M \subseteq T_N$. We will now show that the MI problem for matrices M and N has a negative answer.

The graphs G and H together with a mapping $f : H \xrightarrow{S} G$ are depicted in Fig. 1. The graph G has 4 classes in its degree refinement and H has 14 classes.

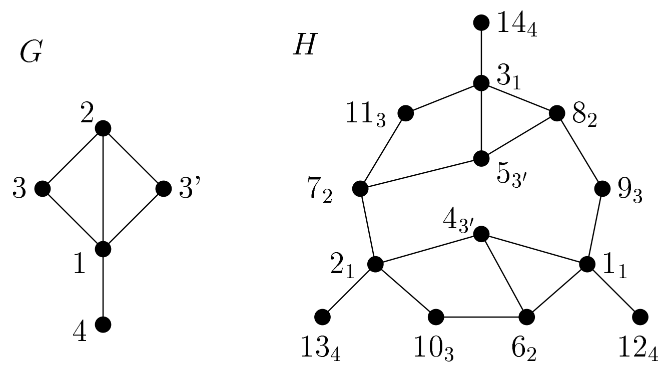

Fig. 1. Graphs G and H, vertices of H are labeled by $u_{f(u)}$ for a $f : H \xrightarrow{S} G$.

Then N is the adjacency matrix of H and the degree refinement matrix of G is

$$M = \begin{pmatrix} 0\,1\,2\,1 \\ 1\,0\,2\,0 \\ 1\,1\,0\,0 \\ 1\,0\,0\,0 \end{pmatrix}.$$

Note that N is the adjacency matrix of H. In order to obtain a contradiction suppose $T_M \xrightarrow{I} T_N$ holds. By Proposition 1 there exist a graph G' with $\mathrm{drm}(G') = M$ and a graph H' with $\mathrm{drm}(H') = N$ such that $G' \subseteq H'$. Let $\{U_1, \ldots, U_k\}$ be the degree partition of G' and $\{V_1, \ldots, V_l\}$ the one for H'. We define the sets W_{rs} as in proof of Theorem 1.

As we have seen in the proof of Theorem 1 the pair (G', H') corresponds with a nontrivial solution of (3). Below we will show, however, that (3) only allows the trivial solution. For simplicity reasons we will first restrict the length of the injective distribution rows.

A vertex in class U_1 has four neighbors in G'. A vertex in class V_4 has three neighbors in H'. This means that a vertex of U_1 can never be in V_4, i.e., $W_{1,4}$ is empty. Hence the set $T(1,4)$ is empty. By the same argument we find that the sets $T(r,s)$ with $(r,s) = (1,5), \ldots, (1,14), (2,9), \ldots, (2,14)$, $(3,12), \ldots, (3,14)$ are empty.

A vertex in U_2 has a neighbor of degree four in G'. A vertex in V_1 does not have a neighbor of degree four in H'. Hence the set $T(2,1)$ is empty. By the same argument we exclude pairs $(2,2), (2,3), (3,1), (3.2), (3,3), (4,1), (4,2), (4,3)$.

Any vertex in U_4 has degree one in G'. Suppose $u \in U_4$ belongs to V_4. So it does not have degree one in H'. Let $v \in U_1$ be the (only) neighbor of u in G'. Then v has degree four in G' and must belong to $V_1 \cup V_2 \cup V_3$. The other three neighbors of v all have degree greater than one in G'. However, one of these three remaining neighbors of v must have degree one in H'. Hence, the set $T(4,4)$ is empty. In the same way we may exclude pairs $(4,4), \ldots, (4,11)$.

Every vertex in $W_{2,4}$ needs a neighbor in $W_{3,1}$ or $W_{3,2}$. These sets are empty, since both $T(3,1)$ and $T(3,2)$ are empty. Hence $T(2,4)$ is empty, and consequently, by a similar argument, $T(3,6)$ is empty. Furthermore, $T(2,4) = \emptyset$ implies that a vertex in $W_{1,2}$ does not have neighbor in $W_{3,7}$. Since every vertex in $W_{3,7}$ must have a neighbor in $W_{1,2}$, the latter implies $T(3,7) = \emptyset$, and consequently $T(2,5) = \emptyset$ and $T(3,8) = \emptyset$.

Only the pairs $(3,4)$ and $(3,5)$ allow two distribution rows, the other pairs all allow one. So we have reduced the total number of feasible distribution rows to $4 \cdot 14 - 20 - 9 - 8 - 5 + 2 = 16$.

The equation (3) for $p, q = 1, 1$ and $i, j = 2, 6$ gives $w^{1,1} = w^{2,6}$. Analogously, $w^{1,1} = w^{3,4(1)}$ while $w^{2,6} = w^{3,4(1)} + w^{3,4(2)}$. Hence $w^{3,4(2)} = 0$. Further $w^{3,4(2)} = w^{1,2} = w^{3,10} = w^{2,6}$, and $w^{1,2} = w^{2,7} = w^{3,11} = w^{1,3}$. Consequently, $w^{1,1} = w^{1,2} = w^{1,3} = 0$.

It can be further shown that (3) allows only trivial solution via values of $w^{r,s}$. However, at this moment we can already claim that no witnesses G, H for $M \xrightarrow{I} N$ exist, since it is impossible to map vertices from the first class of degree partition of G on any vertex of H. $\square$

i	1	1	1	2	2	2	3	3	3	3	3	4	4	4
j	1	2	3	6	7	8	4	5	9	10	11	12	13	14
$p^{1,1}$				1			1		1			1		
$p^{1,2}$					1		1			1			1	
$p^{1,3}$						1		1			1			1
$p^{2,6}$	1						1			1				
$p^{2,7}$		1						1			1			
$p^{2,8}$			1					1	1					
$p^{3,4(1)}$	1			1										
$p^{3,4(2)}$		1		1										
$p^{3,5(1)}$			1		1									
$p^{3,5(2)}$			1			1								
$p^{3,9}$	1					1								
$p^{3,10}$		1		1										
$p^{3,11}$			1		1									
$p^{4,12}$	1													
$p^{4,13}$		1												
$p^{4,14}$			1											

Table 1. The distribution rows for M (only nonzero entries are shown)

4 Matrix comparison via local surjectivity

In this section we are interested in the following matrix comparison problem:

Matrix Surjectivity (MS)
Instance: A degree refinement matrix M and a degree refinement matrix N.
Question: Does $M \xrightarrow{S} N$ hold?

We were not able to answer the decidability of this problem. However, we can show some partial results.

Proposition 3. *Let G be a graph with $\mathrm{drm}(G)$ of order k and H be a graph on l vertices such that $G \xrightarrow{S} H$. Then there exists a graph G' with $\mathrm{drm}(G') = \mathrm{drm}(G)$ such that $G' \xrightarrow{S} H$ and $\langle G' \rangle = (klm^*)^{O(k^2l^2)}$.*

Proof. Let $f : V_G \to V_H$ be a locally surjective homomorphism from G to H. Let $\{U_1, \ldots, U_k\}$ be the degree partition of G and let $\{v_1, \ldots, v_l\}$ be the vertex set of H. We further partition V_G as follows. For each pair of indices r and s we define the set

$$W_{rs} = \{u \mid u \in U_r \text{ and } f(u) = v_s\},$$

and for some vertex $w \in W_{rs}$ we can write a vector describing the distribution of neighbors of w in the classes $W_{11}, \ldots, W_{kl}$.

Let $\mathbf{p}^{rs}$ be a vector of length kl whose entries are positive integers and are indexed by pairs ij. If the vector $\mathbf{p}^{rs}$ further satisfies

$$\sum_{j=1}^{l} p_{ij}^{rs} = m_{ri} \qquad \text{for all } 1 \leq i \leq k, \tag{5}$$

$$(v_s, v_j) \in E_H \qquad \Rightarrow \qquad \sum_{i=1}^{k} p_{ij}^{rs} \geq 1 \qquad \text{for all } 1 \leq j \leq l. \tag{6}$$

$$(v_s, v_j) \notin E_H \qquad \Rightarrow \qquad \sum_{i=1}^{k} p_{ij}^{rs} = 0 \qquad \text{for all } 1 \leq j \leq l. \tag{7}$$

then we call $\mathbf{p}^{rs}$ a *surjective distribution row for indices r and s*. The number of surjective distribution rows is bounded.

We now involve the system of equations (3). We claim that the existence of a nontrivial *nonnegative* solution of (3) is a necessary and sufficient condition for the existence of a finite graph G' with $\mathrm{drm}(G') = M$ and $G \xrightarrow{S} H$. The proof of this claim and the bound on the size of G' is using the same arguments as in the proof of Theorem 1. □

Now we consider the following decision problem.

Matrix Graph Surjectivity (MGS)
Instance: A degree refinement matrix M and a graph H.
Question: Does there exist a graph G with $\mathrm{drm}(G) = M$ such that $G \xrightarrow{S} H$ holds?

Corollary 3. *The* MGS *problem problem is decidable in polynomial space.*

Proof. We can use Proposition 3 and proceed with a proof analogous to the one in Corollary 1. □

We can use Corollary 3 to answer decidability of the MS problem for instances (M, N), where N is the degree refinement matrix of a unique graph H. The proposition below shows that this is only the case if H is a tree.

Proposition 4. *A matrix N is a degree refinement matrix of a unique graph H if and only if N is the degree refinement matrix of a tree.*

Proof. Suppose N is the degree refinement matrix of a tree T. Then the universal cover T_N is isomorphic to T itself. Since all graphs that contain a cycle have an infinite universal cover, there can not be another graph H with $\mathrm{drm}(H) = N$.

In order to prove the reverse statement let H be the only graph that has N as a degree refinement matrix. Suppose H is not a tree. Then H contains an edge $e = (u, v)$ such that the graph $H - e$ is still connected. We take a copy H' of H. Let $e' = (u'v')$ be the copy of e. We remove e in H and e' in H', and we add the edges (u, v') and (u', v). The resulting graph H^* has the same degree refinement matrix as H and is connected. □

We can also use Corollary 3 to answer decidability of the MS problem for instances (M, N), where the $l \times l$ degree refinement matrix N is the adjacency matrix of a graph H. This can be seen as follows. Suppose $M \xrightarrow{S} N$ holds with witnesses G and H'. Since N is an adjacency matrix of graph H, the rows of N are in one-to-one correspondence with vertices of H, i.e., we can say that vertex $v_i \in V_H$ corresponds to row i. Then the function that maps all vertices of H' that belong to block $V_i \subseteq V_{H'}$ to v_i for $1 \leq i \leq l$ is a locally bijective homomorphism from H' to H. The mappings $H' \xrightarrow{B} H$ and $G \xrightarrow{S} H'$ imply $G \xrightarrow{S} H$. So we can restrict ourselves to graph H.

In general, even if we construct a graph G with respect to feasible block sizes, there is no evident rule how to limit the size of some plausible graph H and how to define the locally surjective mapping $G \xrightarrow{S} H$. We leave the general question on decidability of the MS as an open problem.

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