Partitioning Graphs Into Generalized Dominating Sets *

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Abstract

We study the computational complexity of partitioning the vertices of a graph into generalized dominating sets. Generalized dominating sets are parameterized by two sets of nonnegative integers σ and ρ which constrain the neighborhood $N(v)$ of vertices. A set S of vertices of a graph is said to be a (σ, ρ)-set if $\forall v \in S : |N(v) \cap S| \in \sigma$ and $\forall v \notin S : |N(v) \cap S| \in \rho$. The (k, σ, ρ)-partition problem asks for the existence of a partition $V_1, V_2, ..., V_k$ of vertices of a given graph G such that $V_i, i = 1, 2, ..., k$ is a (σ, ρ)-set of G. We study the computational complexity of this problem as the parameters σ, ρ and k vary.

1 Motivation and overview

Several well-studied graph problems ask for a partition of vertices of a graph into subsets with a given property. For example, the Chromatic Number problem asks for a partition into the least number of independent sets. Even the fixed parameter version of this problem, vertex k-coloring, where we ask for the existence of a partition into k independent sets, is intractable (NP-complete) for any $k \geq 3$, while it is easy for smaller k [2]. Similarly, the K_k-cover problem asking for a partition into k perfect codes is NP-complete starting from $k \geq 4$ [5], whereas the partition into perfect matchings problem is NP-complete already for $k \geq 2$ [7].

These problems can all be defined as *Partitions Into Generalized Dominating Sets.* Generalized dominating sets, introduced by Telle in [8] and defined formally in the next section, are parameterized by two sets σ and ρ of nonnegative integers. Many well-studied vertex subset properties with applications in facility location and network communication can be expressed as (σ, ρ)-sets [8, 1, 2, 3], see Table 1. In this paper we present a systematic study of the complexity of partitioning vertices of a given graph G into k (σ, ρ)-sets, for varying values of the parameters k, σ, ρ.

Our focus has been on the vertex subset properties found in the literature, and on generalizations of these properties. In particular, for the vertex subset properties in Table 1, we find the tractability/intractability cutoff point k. Meaning that we give a polynomial-time algorithm deciding whether a graph has a partition into $k-1$ (σ, ρ)-sets, and an NP-completeness proof for the problem of deciding whether it has a partition into k (σ, ρ)-sets. See the table for these cutoff values. Several of our NP-completeness results are shown to hold for infinite classes of problems and/or for restricted graph classes, usually k-regular

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graphs. In addition to the three problems mentioned in the first paragraph, these partition problems have, to our knowledge, previously only been studied for dominating sets (domatic number) [2].

In the next section, we give some definitions and present some general facts on these partitioning problems. Section 3 contains the various NP-completeness reductions. In Section 4, we present the polynomial-time solvable problems and conclude with an open problem in Section 5.

σ	ρ	Standard terminology	cutoff
$\mathbb{N}$	$\{0,1\}$	Nearly Perfect set	∞ *
$\{0\}$	$\{0,1\}$	Strong Stable set or 2-Packing	4 *
$\{0\}$	$\{1\}$	Perfect Code or Efficient Dominating set	4 [5]
$\{0\}$	$\mathbb{N}$	Independent set	3 [2]
$\{0\}$	$\mathbb{P}$	Independent Dominating set	3 *
$\mathbb{N}$	$\mathbb{P}$	Dominating set	3 [2]
$\{0,1\}$	$\{0,1\}$	Total Nearly Perfect set	3 *
$\{0,1\}$	$\{1\}$	Weakly Perfect Dominating set	3 *
$\{1\}$	$\{1\}$	Total Perfect Dominating set	3 *
$\{1\}$	$\mathbb{N}$	Induced Perfect Matching	2 [7]
$\{1\}$	$\mathbb{P}$	Dominating Induced Matching	2 *
$\mathbb{P}$	$\mathbb{P}$	Total Dominating set	2 *
$\mathbb{N}$	$\{1\}$	Perfect Dominating set	2 *

Table 1: Some vertex subset properties expressed as (σ, ρ)-sets, with tractability/intractability cutoff value for the partitioning problem (∞ means tractable for all k values). Reference * if cutoff value proved here.

2 Definitions and some facts

We first define the concepts used in our discussions. They concern mostly combinatorial graphs, *i.e.*, sets of vertices and edges together with the incidence relation between them.

A *graph* $G = (V, E)$ consists of a set of vertices V, and a set of edges $E \subseteq \{(u, v) \mid u, v \in V\}$. We consider only undirected graphs without self-loops. Given a vertex $v \in V$, we call the set of adjacent vertices its *neighborhood*, $N(v) = \{w \mid (v, w) \in E\}$. The cardinality of $N(v)$ is called the *degree* of v. A graph all of whose vertices have degree k is called a *k-regular* graph. A *path* of length k in a graph $G = (V, E)$ is a sequence of distinct, pairwise adjacent vertices, $v_0, v_1, \ldots v_k$, with $(v_{i-1}, v_i) \in E$ for all i, $1 \leq i \leq k$. If, in addition, $(v_0, v_k) \in E$, then we have a *cycle*. A graph is *connected* if there is a path between any two distinct vertices. In the *complete* graph K_k on k vertices there is an edge between every pair of vertices.

Let $\mathbb{N}= \{0, 1, ...\}$ and $\mathbb{P}= \{1, 2, ...\}$ denote the nonnegative and positive integers, respectively. A *partition* $V_1, V_2, ..., V_k$ of a set V satisfies $\bigcup_{i=1}^{k} V_i = V$, and $V_i \cap V_j = \emptyset$, $1 \leq i \neq j \leq k$.

Definition 1 *For $\sigma \subseteq \mathbb{N}$ and $\rho \subseteq \mathbb{N}$, a subset S of the vertices of a graph is a (σ, ρ)-set if $\forall v \in S : |N(v) \cap S| \in \sigma$ and $\forall v \notin S : |N(v) \cap S| \in \rho$.*

Definition 2 *A (k, σ, ρ)-partition of a graph G is a partition $V_1, V_2, ..., V_k$ of its vertices such that each V_i is a (σ, ρ)-set of G for $i = 1, 2, ..., k$.*

Partitioning a graph into (σ, ρ)-sets can also be viewed as a vertex labeling problem. A (k, σ, ρ)-partition of a graph G is a labeling of its vertices with k labels, such that for each vertex, the number of neighbors it has with its own label is in σ and the number of neighbors it has with each different label than its own is in ρ.

We now state some general facts on the existence of (k, σ, ρ)-partitions in a graph G. These facts follow from the definition of a (k, σ, ρ)-partition.

Fact 1 *A graph has a $(1, \sigma, \rho)$-partition if and only if the degree of every vertex is in σ.*

We will be assuming, quite naturally, that the parameter sets σ and ρ have polynomial-time membership tests. The problem of deciding whether a graph has a $(1, \sigma, \rho)$-partition is therefore easy for all values of σ and ρ.

Fact 2 *If p is a positive integer, then in any $(k, \sigma, \{p\})$-partition of a graph G, all partition classes V_i must have the same cardinality $|V_i| = |V(G)|/k$, for $i = 1, ..., k$.*

Fact 2 follows since for any two partition classes V_i and $V_j, i \neq j$, every vertex in V_i must have p neighbors in V_j, and vice-versa.

Fact 3 *For nonnegative integers s and r, if a graph G has a $(k, \{s\}, \{r\})$-partition then G is $(s + (k-1)r)$-regular.*

In the case of Fact 3, every vertex must have exactly s neighbors with the same label as itself, and it must have exactly r neighbors with each of the $k - 1$ other labels.

Fact 4 *Given a graph $G = (V, E)$, if V and $\emptyset$ are both (σ, ρ)-sets, then G has a trivial (k, σ, ρ)-partitioning $V_1 = V$, and $V_i = \emptyset$ for $i = 2, ..., k$.*

We get the situation in Fact 4, for example if $\sigma = \mathbb{N}$ and $0 \in \rho$.

Fact 5 *A graph G has a non-trivial $(k, \sigma, \{0\})$-partition if and only if G contains at least k components, and the degree of every vertex is in σ.*

The problem of deciding whether a graph has a $(k, \sigma, \{0\})$-partition is thus easy for all values of k and σ. The following lemma will be used in the proofs of some of the NP-completeness results of Section 3.

Lemma 6 *For any σ and ρ with $\max \sigma = s$ and $\max \rho = r$, where r and s are nonnegative integers, an $(s + (k-1)r)$-regular graph G has a (k, σ, ρ)-partition if and only if G has a $(k, \{s\}, \{r\})$-partition.*

Proof. If G has a $(k, \{s\}, \{r\})$-partition, then G has clearly a (k, σ, ρ)-partition since $s \in \sigma$, and $r \in \rho$.

If G has a (k, σ, ρ)-partition with $\max \sigma = s$ and $\max \rho = r$, then G has a vertex labeling with k labels such that each vertex has at most s neighbors with the same label as itself and at most r neighbors labeled with each of the $k - 1$ other labels. Since G is $(s + (k-1)r)$-regular, each vertex v has exactly $s + (k-1)r$ neighbors, where s neighbors must be labeled with the same label as v, and r neighbors must be labeled with each of the other $k-1$ labels. This is equivalent to a $(k, \{s\}, \{r\})$-partition of G. ■

For all our results it will turn out that tractability of a (k, σ, ρ)-partition problem holds even if k is part of the input, while an intractability result always holds for k fixed. This is not surprising, as the number of partitions of n vertices into k classes is not polynomial in n for any $k \geq 2$.

3 NP-completeness results

The values of σ and ρ used to describe the vertex subset properties listed in Table 1 are confined to $\{0\}, \{0,1\}, \{1\}, \mathbb{N}$ and $\mathbb{P}$. For $\rho = \{0\}$ we know from Fact 5 that the corresponding partition problems are easy. Table 2 gives a complete listing of all remaining combinations of σ and ρ among these values. We focus our study on the problems in this table. In this section, we give the NP-completeness proofs for the following (k, σ, ρ)-partition problems, which have not been proven before:

1. $(2, \mathbb{N}, \{1\})$, $(2, \mathbb{P}, \{1\})$, $(2, \{0,1\}, \mathbb{N})$ $(2, \{0,1\}, \mathbb{P})$, $(2, \mathbb{P}, \mathbb{P})$ and $(2, \{1\}, \mathbb{P})$.
2. $(k, \{1\}, \{1\})$, $(k, \{1\}, \{0,1\})$, $(k, \{0,1\}, \{1\})$, $(k, \{0,1\}, \{0,1\})$, $(k+1, \{0\}, \{0,1\})$ all on k-regular graphs, and $(k, \{0\}, \mathbb{P})$, all for $k \geq 3$.

These problems are marked with a * in our tables. The entries in Table 2 are the cutoff values for the corresponding (k, σ, ρ)-partition problems. In order to show that these k-values are the cutoff points, polynomial time algorithms are given in Section 4 for partitioning into less than k (σ, ρ)-sets.

σ \ ρ	$\mathbb{N}$	$\mathbb{P}$	$\{1\}$	$\{0,1\}$
$\mathbb{N}$	∞^*	3	2^*	∞^*
$\mathbb{P}$	∞^*	2^*	2^*	∞^*
$\{1\}$	2	2^*	3^*	3^*
$\{0,1\}$	2^*	2^*	3^*	3^*
$\{0\}$	3	3^*	4	4^*

Table 2: A two dimensional cutoff table.

Before the NP-completeness proofs, we give definitions of three problems which do not appear in previous sections. These problems are all known to be NP-complete, and are used in the proofs of some of the theorems of this subsection.

Definition 3 (1-3SAT) *Given sets $S_1, ..., S_m$ each having three members, the One-In-Three Satisfiability problem asks whether there is a subset T of the members such that for each i, $|T \cap S_i| = 1$.*

Definition 4 (NAE-3SAT) *Given a collection C of clauses on a finite set X of variables where each clause contains three literals, the Not-All-Equal 3-Satisfiability problem asks whether there is a truth assignment for X that satisfies all the clauses in C such that no clause contains literals that are all true.*

The problems 1-3SAT and NAE-3SAT were shown to be NP-complete by Schaefer in [7].

Definition 5 (k-EC) *Given a graph G, the k-Edge-Coloring problem asks whether the edges of G can be colored with k colors such that no vertex is incident to two edges colored with the same color. If G is a k-regular graph, the question becomes whether each vertex is incident to k distinctly colored edges.*

This last problem was shown to be NP-complete for $k = 3$ by Holyer in [4], and for $k \geq 3$ by Leven and Galil in [6].

We now give the proofs of the NP-completeness results for the partition problems listed at the beginning of Section 3. Clearly all of these problems are in NP, and as will be obvious, all of our reductions are polynomial-time reductions.

Theorem 1 *The following (k, σ, ρ)-partition problems are NP-complete:*

(i) $(2, \mathbb{N}, \{1\})$ - *Perfect Matching Cut*

(ii) $(2, \mathbb{P}, \{1\})$ - *Partition Into 2 Perfect Dominating Sets Inducing No Isolated Vertices.*

Proof. **(i)** We use a reduction from 1-3SAT to $(2, \mathbb{N}, \{1\})$-partition. Let us consider an arbitrary instance of 1-3SAT, where $S_1, S_2, ..., S_m$ are the given sets, each containing three variables. We construct a graph $G = (V, E)$ such that G is $(2, \mathbb{N}, \{1\})$-partitionable if and only if the sets $S_1, S_2, ..., S_m$ are one-in-three satisfiable.

For each set $S_i = \{x, y, z\}$, we construct a subgraph G_i of G consisting of 10 vertices $x_i, y_i, z_i, a_i^1, a_i^2, ..., a_i^7$, as shown in Figure 1. Whenever two sets S_i and S_j share a variable x, there is an edge between x_i and x_j in G. G also contains a 4-cycle $[s, t_2, t_1, t_3, s]$ induced by the vertices s, t_1, t_2 and t_3 with $(s, a_i^1) \in E$, for $i = 1, ..., m$ (t_1, t_2, t_3 and s have no other neighbors). An example is shown in Figure 2.

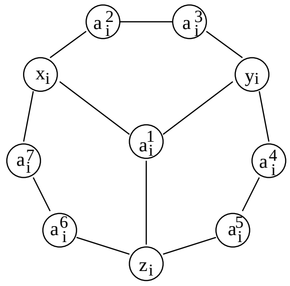

Figure 1: The subgraph of G corresponding to $S_i = \{x, y, x\}$.

Let G be $(2, \mathbb{N}, \{1\})$-partitionable. This is equivalent to a 2-labeling of the vertices of G such that each vertex has exactly one neighbor with the opposite label than itself. The vertices t_2 and t_3 must be labeled oppositely since they are the only neighbors of t_1. Equivalently, s and t_1 must also be labeled oppositely. Assume, without loss of generality, that t_1 is labeled with 1 and s with 0. Since s already has a neighbor (t_2 or t_3) with the opposite label, every other neighbor of s must be labeled with the same label as s. Thus a_i^1 is labeled with 0, for $i = 1, ..., m$. Consider a subgraph G_i, where $S_i = \{x, y, z\}$ is the

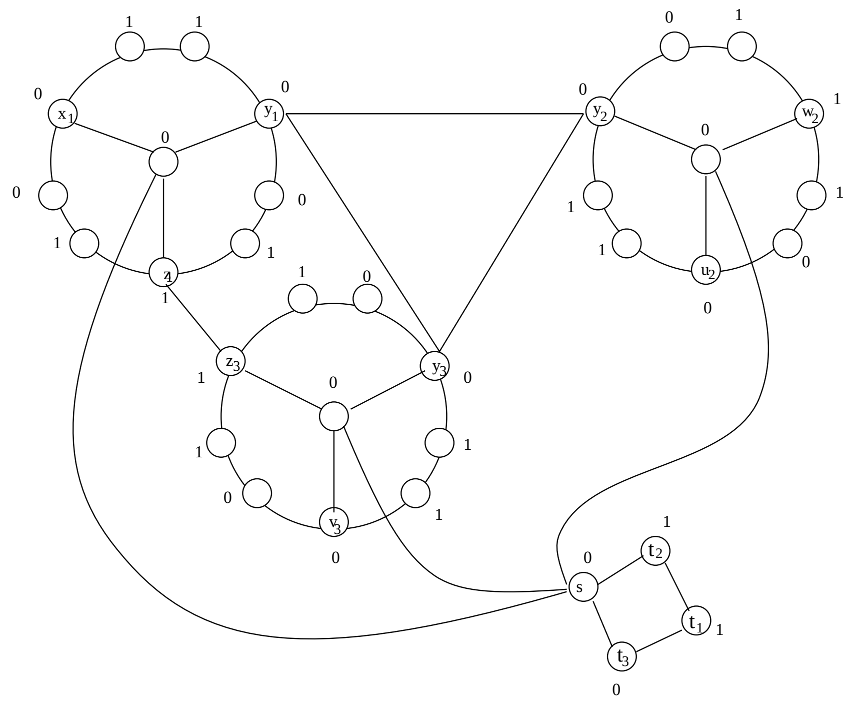

Figure 2: An example showing the constructed graph G, when the given sets are: $S_1 = \{x, y, z\}, S_2 = \{u, w, y\}$ and $S_3 = \{v, y, z\}$. The labeling corresponds to choosing $T = \{w, z\}$.

corresponding set. Apart from s, a_i^1 has neighbors x_i, y_i, z_i. Since both s and a_i^1 are labeled with 0, and a_i^1 must have exactly one neighbor with the opposite label, one of x_i, y_i and z_i must be labeled with 1, and the other two with 0. When these are labeled, the rest of the labeling of G_i is unique. This labeling is such that each of x_i, y_i and z_i has exactly one neighbor with the opposite label than itself within G_i. Therefore, any neighbor outside G_i of a vertex x_i must have the same label as x_i. The only such neighbors of x_i are the vertices x_j, where $x \in S_i \cap S_j$. If G has a valid labeling, then the desired set T for 1-3SAT is $T = \bigcup_{i=1}^{m} \{x \in S_i \mid x_i \text{ is labeled with } 1\}$. Now, each set S_i has exactly one member x such that the corresponding vertex x_i in G_i is labeled with 1. Also, for each other set S_j containing x, the corresponding vertex x_j must also be labeled with 1. Thus the sets $S_1, ..., S_m$ are one-in-three satisfiable.

Assume now that there exists a set $T \subset \bigcup_{i=1}^{m} S_i$, such that each set S_i has exactly one member in T. For each $x \in T$, and for each i such that $x \in S_i$, label the corresponding vertex x_i in G with label 1. For each $y \notin T$, and for each i such that $y \in S_i$, label the corresponding vertex y_i with 0. Label t_2 and t_3 oppositely, t_1 with 1, s with 0, and a_i^1 with 0, for $i = 1, ..., m$. Let us now consider a subgraph G_i corresponding to $S_i = \{x, y, z\}$. Assume that $z \in T$. Label a_i^2, a_i^3, a_i^5 and a_i^6 with label 1, and a_i^4 and a_i^7 with 0. This describes a valid labeling of all the vertices in G_i. Because of the symmetry of G_i, this labeling can easily be adapted for $x \in T$, or $y \in T$, instead of z (see Figure 2). Let us label all the subgraphs G_i as described. Since each subgraph has a valid labeling independent of the rest of G, and since external edges only go between vertices of same label, this describes a valid labeling of G. Thus G is $(2, \mathbb{N}, \{1\})$- partitionable.

(ii) Note that in G, every vertex has at least two neighbors. Therefore, G has a $(2, \mathbb{N}, \{1\})$-partitioning if and only if it has a $(2, \mathbb{P}, \{1\})$-partitioning. It is easy to see that the labeling described in **(i)** is indeed a $(2, \mathbb{P}, \{1\})$-partitioning for G. ■

Theorem 2 *The following (k, σ, ρ)-partition problems are NP-complete:*

(i) $(2, \{0, 1\}, \mathbb{N})$ - *Partition Into 2 Sets Inducing Isolated Vertices and Edges.*

(ii) $(2, \{0, 1\}, \mathbb{P})$ - *Partition Into 2 Dominating Sets Inducing Isolated Vertices and Edges.*

Proof. **(i)** The reduction is from 1-3SAT to $(2, \{0, 1\}, \mathbb{N})$-partition. Let $S_1, S_2, ..., S_m$ be the given sets in an arbitrary instance of 1-3SAT. We construct a graph $G = (V, E)$ such that G is $(2, \{0, 1\}, \mathbb{N})$-partitionable if and only if the sets $S_1, S_2, ..., S_m$ are one-in-three satisfiable.

For each set $S_i = \{x, y, z\}$, there is a 4-clique K_i in G induced by the vertices x_i, y_i, z_i and a_i, as shown in Figure 3 a). For each variable x, there is an edge e_x in G. For each set S_i such that $x \in S_i$, the vertex x_i of K_i has edges to both endpoints of e_x, as shown in Figure 3 b). G also contains a 4-clique induced by the vertices s, t_1, t_2 and t_3 with $(s, a_i) \in E$, for $i = 1, ..., m$ (t_1, t_2, t_3 and s have no other neighbors). An example is shown in Figure 4.

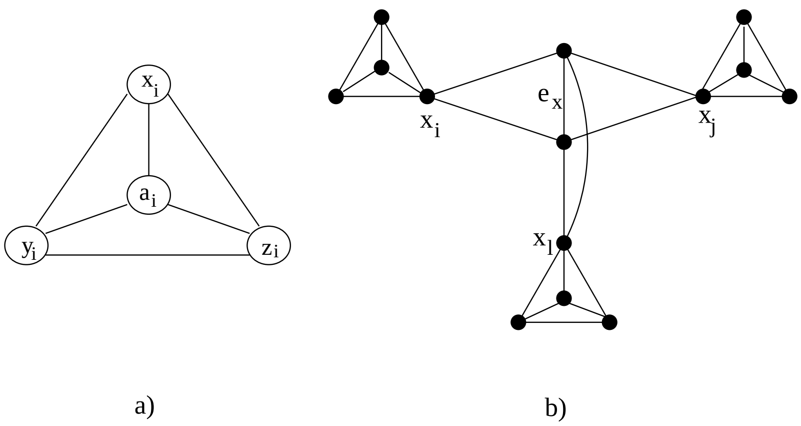

Figure 3: a) The clique K_i corresponding to the set $S_i = \{x, y, z\}$, and b) the subgraph concerning e_x when $x \in S_i \cap S_j \cap S_l$.

A $(2, \{0, 1\}, \mathbb{N})$-partitioning of G is equivalent to a 2-labeling of the vertices such that each vertex has at most one neighbor with the same label as itself. Assume that G has such a labeling. Each 4-clique must be labeled such that two vertices are labeled with 0 and two are labeled with 1, otherwise some vertex in the clique would have two or more neighbors that are labeled the same as itself. Assume without loss of generality that s is labeled with 0. Then two of t_1, t_2 and t_3 are labeled with 1, and one with 0. Since s already has a neighbor with the same label as itself, every other neighbor of s must be labeled with the opposite label. Thus each a_i is labeled with 1, for $i = 1, 2, ..., m$. Let K_i be the 4-clique corresponding to the set $S_i = \{x, y, z\}$. Since K_i must have exactly two vertices with each of the labels, one of x_i, y_i and z_i must be labeled with 1, and two must be labeled with 0. Each vertex x_i has now one neighbor in K_i with the same label as x_i. Therefore, every other

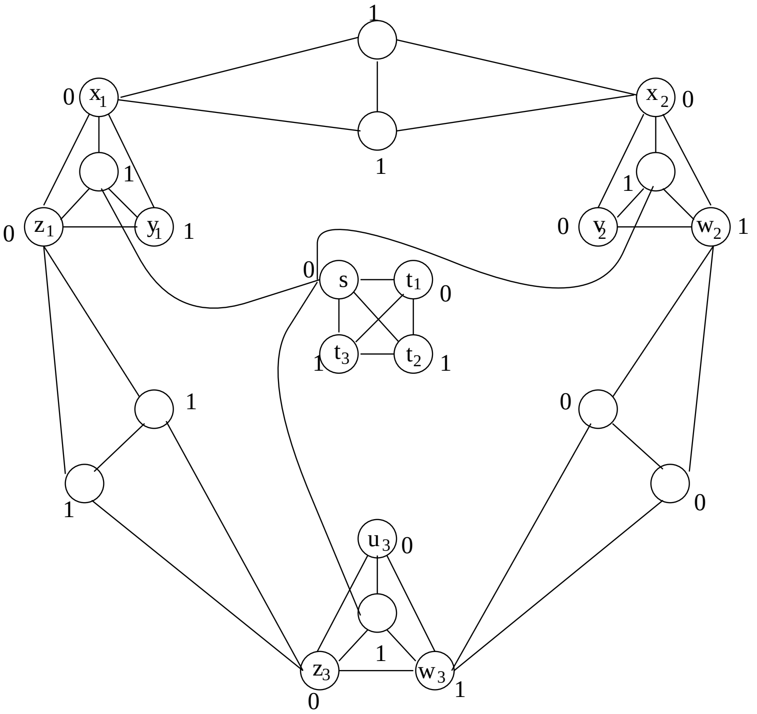

Figure 4: An example showing the constructed graph G, when the given sets are: $S_1 = \{x, y, z\}, S_2 = \{x, v, w\}$ and $S_3 = \{u, w, z\}$. The labeling corresponds to choosing $T = \{w, y\}$.

neighbor of x_i must be labeled with the opposite label. This implies that both endpoints of the edge e_x have the same label, and every other neighbor of these endpoints must be labeled with the same label as x_i. Now, the desired set T in 1-3SAT is $T = \bigcup_{i=1}^{m}\{x \in S_i \mid x_i$ is labeled with $1\}$, and we can conclude that the sets $S_1, ..., S_m$ are one-in-three satisfiable.

Assume that the sets $S_1, ..., S_m$ are one-in-three satisfiable, and let T be the desired subset. Label a_i with 1, for $i = 1, 2, ..., m$, and label rest of the vertices in each clique K_i with 1 if the corresponding variables belong to T, and with 0 otherwise. Since there is only one member in each set that can have its corresponding vertex labeled with 1, there are two vertices labeled with 1, and two with 0 in each K_i. Label s and t_1 with 0, and t_2 and t_3 with 1. For each edge e_x, label both endpoints with 0 if $x \in T$, and with 1 otherwise. Clearly, this is a labeling of G corresponding to a $(2, \{0, 1\}, \mathbb{N})$-partition.

(ii) Since each vertex in G has at least two neighbors, a $(2, \{0, 1\}, \mathbb{N})$-partition for G is equivalent to a $(2, \{0, 1\}, \mathbb{P})$-partition. The labeling described in **(i)** is indeed a $(2, \{0, 1\}, \mathbb{P})$-partitioning of G. ■

Theorem 3 *The* $(2, \mathbb{P}, \mathbb{P})$*-partition problem (Partition Into Two Total Dominating Sets) is NP-complete.*

Proof. The reduction is from NAE-3SAT. Let X be the set of variables and C be the the set of clauses in an arbitrary instance of NAE-3SAT. We can assume that all literals appear in some clause, otherwise for each literal that does not appear, we can put it and its negation in an additional clause. This clause is then always true and contains at least one false literal. We construct a graph $G = (V, E)$ such that G has a $(2, \mathbb{P}, \mathbb{P})$-partition if and only if the variables in X can be assigned values true or false such that each clause in C has at least one literal that is true and at least one that is false.

For each variable $x \in X$, there are three vertices x, $\bar{x}$ and a_x in V, where $(x, \bar{x}) \notin E$, $(x, a_x) \in E$, and $(\bar{x}, a_x) \in E$. For each clause $c_i \in C$, there is a vertex $c_i \in V$, and c_i has edges to the vertices representing the literals in the clause c_i. This construction is shown in Figure 5.

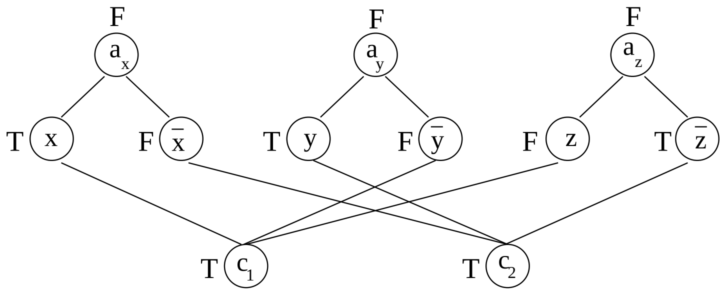

Figure 5: An example showing G when the given clauses are: $c_1 = (x, \bar{y}, z)$ and $c_2 = (\bar{x}, y, \bar{z})$.

A $(2, \mathbb{P}, \mathbb{P})$-partitioning of G is a 2-labeling of the vertices of G such that each vertex $v \in V$ has at least one neighbor with the same label as v, and at least one neighbor with the opposite label. Assume that G has such a labeling, and let the two labels be T and F. Since a_x has only two neighbors x and $\bar{x}$, the vertices x and $\bar{x}$ have opposite labels for each $x \in X$. Furthermore, each vertex c_i has at least one neighbor labeled with T, and at least one neighbor labeled with F. With $T =$ true and $F =$ false, this describes a truth assignment for the variables in X that is consistent with the requirements of the NAE-3SAT problem. An example labeling is given in Figure 5.

Conversely, given that X has such a truth assignment, we will show how to achieve a 2-labeling of the vertices of G that corresponds to a $(2, \mathbb{P}, \mathbb{P})$-partitioning. Label each vertex a_x with F, for $x \in X$, and label each vertex c_i with T, for $c_i \in C$. Each vertex x and $\bar{x}$ is labeled with T if its corresponding literal is assigned true, and with F if its corresponding literal is assigned false. Clearly, this describes a valid $(2, \mathbb{P}, \mathbb{P})$-partitioning of G, and the proof is complete. ∎

Theorem 4 *The* $(2, \{1\}, \mathbb{P})$*-partition problem (Partition Into 2 Dominating Induced Matchings) is NP-complete on 3-regular graphs.*

Proof. The reduction is from the $(2, \{1\}, \mathbb{N})$-partition problem, which is also called *Two-Colorable Perfect Matching.* Schaefer shows in [7] that this problem is NP-complete on 3-regular graphs.

Let G be a $(2, \{1\}, \mathbb{N})$-partitionable 3-regular graph. Then the vertices of G can be labeled with two labels such that each vertex has exactly one neighbor labeled with the same label as itself. Since G is 3-regular, this also means that each vertex has exactly two neighbors labeled with the opposite label as itself. Thus G is also $(2, \{1\}, \mathbb{P})$-partitionable.

Conversely, a $(2, \{1\}, \mathbb{P})$-partition for any graph is clearly a $(2, \{1\}, \mathbb{N})$-partition for the same graph. Thus, a 3-regular graph G is $(2, \{1\}, \mathbb{N})$-partitionable if and only if G

is $(2, \{1\}, \mathbb{P})$-partitionable, and we have shown that $(2, \{1\}, \mathbb{P})$-partition problem is NP-complete on 3-regular graphs. ■

Theorem 5 *The $(k, \{1\}, \{1\})$-partition problem (Partition Into k Total Perfect Dominating Sets) is NP-complete on k-regular graphs for $k \geq 3$.*

Proof. The reduction is from k-EC on k-regular graphs for $k \geq 3$. Let $G = (V, E)$ be a k-regular graph. We construct a k-regular graph $G' = (V', E')$ such that G' has a $(k, \{1\}, \{1\})$-partitioning if and only if G has a k-edge-coloring.

For each vertex $v \in V$, there is a clique K_v in G' with k vertices, and $V' = \bigcup_{v \in V} K_v$. Each vertex of K_v has exactly one neighbor outside K_v corresponding to the edges incident to v in G. If $(u, v) \in E$, then in E', there is an edge between exactly one vertex of K_u and exactly one vertex of K_v. This construction is shown in Figure 6. Since G is k-regular, each clique K_v in G' has exactly k external edges incident to it, one incident to each of the k vertices in K_v. Thus G' is also k-regular. Any two cliques K_v and K_w in G' have at most one external edge connecting them.

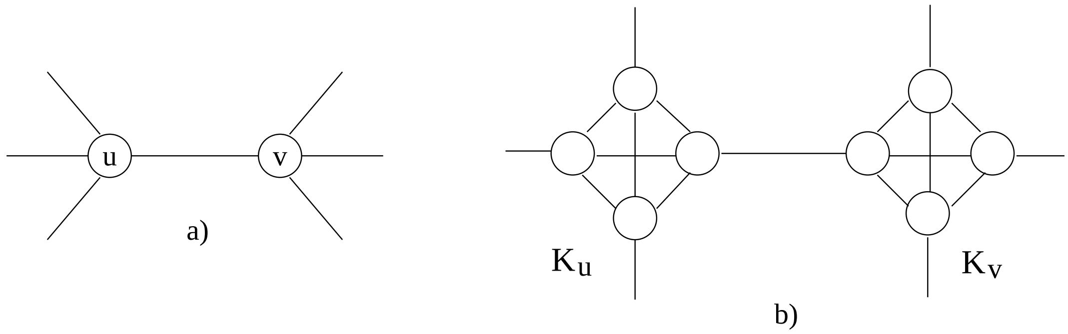

Figure 6: a) An edge in G, and b) the corresponding subgraph in G', where G is 4-regular.

A $(k, \{1\}, \{1\})$-partitioning of a graph is equivalent to a k-labeling of the vertices such that each vertex has exactly k neighbors each labeled distinctly with one of the k labels (one neighbor of its own label, and exactly $k - 1$ other neighbors each labeled distinctly). Assume that G' has a k-labeling. The internal vertices of each clique K_v must be labeled with k different labels, otherwise some of the vertices in the clique would have at least two neighbors labeled with the same label. In addition, both endpoints of each external edge must be labeled with the same label, since each endpoint z has $k - 1$ other neighbors labeled distinctly and differing from the label of z in the clique that z belongs to. An example is shown in Figure 7. For each external edge in G', if the endpoints are labeled with label i, then color the corresponding edge in G with color i. Since the vertices in each clique are labeled distinctly in G', each vertex in G will then be incident to k distinctly colored edges, and G is k-edge-colorable.

If G is k-edge-colorable with colors $\{1, 2, ..., k\}$, then there exists a k-labeling of G' with labels $\{1, 2, ..., k\}$, such that for each edge colored with i in G, there is an external edge in G' both of whose endpoints are labeled with i, and where the internal vertices of each clique K_v are labeled with k distinct labels. Thus G' is $(k, \{1\}, \{1\})$-partitionable, and the proof is complete. ■

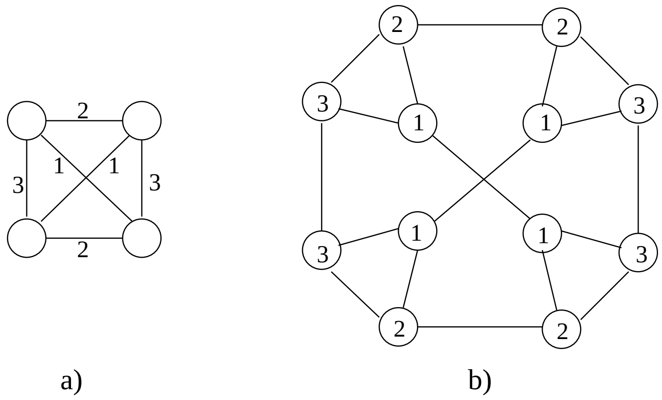

Figure 7: a) A 3-regular graph G which is 3-edge-colorable, and b) the vertex labeling of the corresponding graph G'.

Theorem 6 *The following (k, σ, ρ)-partition problems are NP-complete on k-regular graphs for $k \geq 3$:*

(i) $(k, \{0, 1\}, \{1\})$ - *Partition Into k Weakly Perfect Dominating Sets*

(ii) $(k, \{0, 1\}, \{0, 1\})$ - *Partition Into k Total Nearly Perfect Set s*

(iii) $(k, \{1\}, \{0, 1\})$ - *Partition Into k Nearly Perfect Sets Inducing Perfect Matching.*

Proof. Let $s = \max \sigma$, and $r = \max \rho$. For all these three problems, $s = r = 1$, and $s + (k-1)r = k$. By Lemma 6, a k-regular graph G has a $(k, \{1\}, \{1\})$-partitioning if and only if it has a (k, σ, ρ)-partitioning where $\sigma \in \{\{0, 1\}, \{1\}\}$, and $\rho \in \{\{0, 1\}, \{1\}\}$. Thus Theorem 5 implies that all of these problems are NP-complete on k-regular graphs for $k \geq 3$. ■

Theorem 7 *The $(k, \{0\}, \{0, 1\})$-partition problem (Partition Into k Strong Stable Sets) is NP-complete on $(k-1)$-regular graphs for $k \geq 4$.*

Proof. The reduction is from the $(k, \{0\}, \{1\})$-partition problem, which is also called *Partition Into k Perfect Codes* . Kratochvil shows in [5] that this problem is NP-complete on $(k-1)$-regular graphs for $k \geq 4$. Let $s = \max \sigma$ and $r = \max \rho$. For these two problems, $s = 0$, $r = 1$, and $s + (k-1)r = k - 1$. Then by Lemma 6, a $(k-1)$-regular graph G has a $(k, \{0\}, \{1\})$-partitioning if and only if G has a $(k, \{0\}, \{0, 1\})$-partitioning. ■

Theorem 8 *The $(k, \{0\}, \mathbb{P})$-partition problem (Partition Into k Independent Dominating Sets) is NP-complete for $k \geq 3$.*

Proof. The reduction is again from k-EC on k-regular graphs. Given a k-regular graph $G = (V, E)$, we construct a graph G' such that G' has a $(k, \{0\}, \mathbb{P})$-partitioning if and only if G is k-edge-colorable.

For each vertex $v \in V$, there is a k-clique K_v in G', and for each edge $(u, v) \in E$, there is a $(k-1)$-clique E_{uv} in G'. Let $(u, v) \in E$, then in G', there is exactly one vertex in K_u,

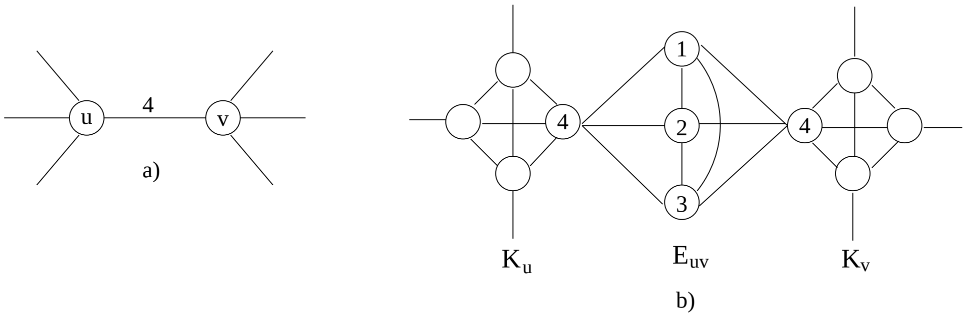

Figure 8: a) An edge in G, and b) the corresponding subgraph of G', where G is 4-regular.

and exactly one vertex in K_v that have edges to every vertex in E_{uv}, and no other vertex in G' has edges to the vertices in E_{uv}. This is depicted in Figure 8.

Now, a $(k, \{0\}, \mathbb{P})$-partitioning of a graph is equivalent to a k-labeling of the vertices such that each vertex has no neighbors labeled the same as itself, and at least one neighbor labeled with each of the other $k-1$ labels. Vertices in each k-clique must be labeled with k distinct labels, otherwise at least one vertex v in the cliques would have neighbors that are labeled the same as v. Since each vertex of G' appears in a k-clique, this is equivalent to a proper k-coloring of the vertices of G'. Consider Figure 8; if the vertex connecting K_u to E_{uv} is labeled with i, then the vertex connecting E_{uv} to K_v must also be labeled with i since the $(k-1)$-clique in between requires all of the remaining $k-1$ labels. In G, we can color the edge (u, v) with color i. With an argument equivalent to the one in the proof of Theorem 5, we can conclude that G is k-edge-colorable if and only if G' is $(k, \{0\}, \mathbb{P})$-partitionable. ■

4 Polynomial time results

The polynomial-time solvable (k, σ, ρ)-partition problems listed in Table 2 fall in four classes, corresponding to a cutoff value of 2, 3, 4 or ∞. For each (σ, ρ)-set we show that deciding whether a graph has a partition into k classes, where k is less than the cutoff value, is easy.

For the problems with cutoff value 2 this indeed follows as stated in Fact 1. The problems with cutoff value ∞ either have the property that a trivial partition with $V_1 = V$ and $V_i = \emptyset, i = 1, ..., k$ is a k-partition (see Fact 4) or they have $\rho = \{0\}$ and are easy by Fact 5.

For the problems with cutoff value 3 or 4, the properties that must be checked to decide whether a graph has a partition of size smaller than the cutoff value are summerized in Table 3. We explain the first two of these problems and their solutions further, whereas we leave the rest to be studied in the same way by the reader. Here we assume that the given graph is connected, if it is not, the same argument can be applied on each connected component.

Let us first consider the $(2, \{1\}, \{1\})$-partition problem. This is equivalent to finding a 2-labeling of the vertices of the given graph G such that each vertex v has exactly one neighbor labeled with the same label as v, and exactly one neighbor labeled with the opposite label. Since the degree of each vertex in G must then be 2, G must be a cycle. The desired labeling is possible if and only if the number of vertices in the cycle is a multiple of 4. Thus we can

(k,σ,ρ)	(k,σ,ρ)-partitionable ($m \geq 1$)
$(2,\{1\},\{1\})$	C_{4m}
$(2,\{1\},\{0,1\})$	$\{P_{2m-1}, C_{4m}\}$
$(2,\{0,1\},\{1\})$	$\{P_{2m-1}, C_{4m}\}$
$(2,\{0,1\},\{0,1\})$	$\{P_m, C_{4m}\}$
$(2,\{0\},\mathbb{P})$	Bipartite
$(3,\{0\},\{0,1\})$	$\{P_m, C_{3m}\}$

Table 3: For each row, G has the (k,σ,ρ)-partition on the left if and only if G or its connected components belong to the class on the right.

check in polynomial time whether G is a $4m$-cycle.

The $(2,\{1\},\{0,1\})$-partition problem asks for a 2-labeling of the given graph G such that each vertex v has exactly one neighbor labeled with the same label as v, and at most one neighbor labeled with the opposite label. Since the degree of each vertex in G must then be at most 2, G must be either a cycle or a path. If G is a cycle, then it must be a $4m$-cycle as above. Otherwise, G must be a path, and since each vertex has exactly one neighbor with the same label as itself, there must be an even number of vertices in the path. Thus we must check whether $G \in \{P_{2m-1}, C_{4m}\}$ for $m \geq 1$.

5 Closing remarks

We have presented a systematic study of the complexity of partitioning graphs into generalized dominating sets, with emphasis on those generalized dominating sets which are found in the literature. If we look more broadly at the general class of (k,σ,ρ)-partition problems, we believe that they will all fall in one of the 4 classes corresponding to a cutoff value of 2,3,4 or ∞. We may mention that for graphs of bounded treewidth there are linear-time algorithms solving any (k,σ,ρ)-partition problem [9].

Most problems with cutoff 3 or 4 are easily solvable for partitions into 2 or 3 classes either because such partitions exist only if the graph has maximum degree 2, or because $\sigma = \{0\}$ and a partition into 2 classes exists if and only if the graph is bipartite and the degree of every vertex is in ρ. The exception to this rule is the $(2,\mathbb{N},\mathbb{P})$ problem, partitioning a graph into two dominating sets, which can always be done by taking one subset to be a maximal independent set, easily found by a greedy algorithm.

For problems with cutoff ∞, it might be of interest to consider only non-trivial partitions, *i.e.*, allow only non-empty partition classes. Some problems, like $(k,\mathbb{N},\mathbb{N})$ are still easy, since every subset of vertices in a graph is a $(\mathbb{N},\mathbb{N})$-set.

Let us look at the $(k,\mathbb{P},\mathbb{N})$-partition problem into non-empty classes $V_1,...,V_k$ and show that it can be solved for all k using a maximum matching of the graph G. Clearly, G cannot have any isolated vertices. Let $M = \{e_1, e_2, ..., e_m\}$ be a maximum matching of G. If $m \geq k$, then we place in V_i the endpoints of the edge e_i, for $i = 1,...,k-1$. The endpoints of the edges $e_k, e_{k+1}, ..., e_m$ are placed in V_k. Since there are no isolated vertices in G, every other vertex in the graph must be a neighbor of some of the vertices that are already placed in the partition sets. For each such vertex v, place v in any one of the partition sets that contains a neighbor of v. Now, all the vertices in G are partitioned into the sets $V_1,...,V_k$, and each set contains at least two vertices where each vertex has at least one neighbor in the same set as itself. Note also that, given a non-trivial $(k,\mathbb{P},\mathbb{N})$-partition for G, a matching with

k or more edges can be found by picking an edge from each partition set. Thus G has a non-trivial $(k, \mathbb{P}, \mathbb{N})$-partitioning if and only if G has a matching with k or more edges. Since a maximum matching can be found in polynomial time, this describes a polynomial time algorithm for the (non-trivial) $(k, \mathbb{P}, \mathbb{N})$-partition problem.

We have not been able to resolve the complexity of the (non-trivial) $(k, \mathbb{N}, \{0,1\})$ and $(k, \mathbb{P}, \{0,1\})$-partition problems and will therefore close the paper with the the following open problem.

Open Problem: What is the complexity of deciding whether the vertices of a given graph G can be labeled by two labels such that each vertex has at most one neighbor of a different label than its own, and there is at least one vertex with each label?

References

[1] E. J. Cockayne, B. L. Hartnell, S. T. Hedetniemi, and R. Laskar, *Perfect domination in graphs*, J. Combinatorics Inform. Systems Sci., 18 (1993), pp. 136–148.

[2] M. R. Garey and D. S. Johnson, *Computers and Intractability*, W. H. Freeman and Co., 1978.

[3] S. T. Hedetniemi and R. Laskar, *Bibliography on domination in graphs and some basic definitions of domination parameters*, Discrete Mathematics, 86 (1990).

[4] I. Holyer, *The NP-completeness of edge-coloring*, Siam Journal of Computing, 10 (1981), pp. 718–720.

[5] J. Kratochvil, *Perfect Codes in General Graphs*, Academia Praha, 1991.

[6] D. Leven and Z. Galil, *NP-completeness of finding the chromatic index of regular graphs*, Journal of Algorithms, 4 (1983), pp. 35 – 44.

[7] T. J. Schaefer, *The complexity of satisfiability problems*, ACM Symposium on Theory of Computing, 10 (1978), pp. 216–226.

[8] J. A. Telle, *Complexity of domination-type problems in graphs*, Nordic Journal of Computing, 1 (1994), pp. 157–171.

[9] ——, *Vertex Partitioning Problems: Characterization, Complexity and Algorithms on Partial k-Trees*, PhD thesis, University of Oregon, 1994.