Computational Complexity of Covering Three-Vertex Multigraphs

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Abstract. A covering projection from a graph G to a graph H is a mapping of the vertices of G to the vertices of H such that, for every vertex v of G, the neighborhood of v is mapped bijectively to the neighborhood of its image. Moreover, if G and H are multigraphs, then this local bijection has to preserve multiplicities of the neighbors as well. The notion of covering projection stems from topology, but has found applications in areas such as the theory of local computation and construction of highly symmetric graphs. It provides a restrictive variant of the constraint satisfaction problem with additional symmetry constraints on the behavior of the homomorphisms of the structures involved.

We investigate the computational complexity of the problem of deciding the existence of a covering projection from an input graph G to a fixed target graph H. Among other partial results this problem has been shown to be NP-hard for simple regular graphs H of valency greater than 2, and a full characterization of computational complexity has been shown for target multigraphs with 2 vertices. We extend the previously known results to the ternary case, i.e., we give a full characterization of the computational complexity in the case of multigraphs with 3 vertices. We show that even in this case a P/NP-completeness dichotomy holds.

Keywords: Computational Complexity, Graph Homomorphism, Covering Projection

1 Introduction

The concept of covering spaces or covering projections stems from topology, but has attracted a lot of attention in algebra, combinatorics, and also the theory of computation. For instance, it is used in algebraic graph theory as a very

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useful tool for the construction of highly symmetric graphs. The applications in computability include the theory of local computations (cf. [2] and [7]). A lot of interest has been paid to graphs that allow finite planar covers. This class of graphs is closed in the minor order and hence recognizable in polynomial time, yet despite a lot of effort no concrete recognition algorithm is known, since the obstruction set has not been determined yet. The class has been conjectured to be equal to the class of projective planar graphs by Negami [19] (for the most recent results cf. [11, 12]).

In [1], Abello et al. raised another complexity question, asking about the computational complexity of deciding the existence of a covering projection from an input graph G to a fixed graph H (hoping for a characterization giving a P/NP-completeness dichotomy depending on H). A similar question when both G and H are part of the input was shown NP-complete by Bodlaender already in 1989 [4]. The dichotomy asked for by Abello et al. seems to be hard to obtain and only very partial results are known. The most general NP-completeness result states that for every simple regular graph H of valency at least 3, the problem is NP-complete [17]. No plausible conjecture on the borderline between polynomially solvable and NP-complete instances has been published so far, yet it is believed that a P/NP-completeness dichotomy will hold, as in the case of the constraint satisfaction problem (CSP).

The relation to CSP is worth mentioning in more detail. As shown in [9], for every fixed graph H, the H-Cover problem can be reduced to CSP, but mostly to NP-complete cases of CSP, so this reduction does not help. In a sense a covering projection is itself a variant of CSP, but with further constraints of local symmetry. Thus the dichotomy conjecture for H-Cover does not follow from the well-known Feder-Vardi dichotomy conjecture for CSP (cf. [8]).

In [16] it is shown that in order to fully understand the H-Cover problem for simple graphs, one has to understand its generalization for colored mixed multigraphs. For this reason we are dealing with multigraphs (undirected) in this paper. Kratochvil et al. [16] completely characterized the computational complexity of the H-Cover problem for colored mixed multigraphs on two vertices. The aim of this paper is to extend this characterization to 3-vertex multigraphs (in the undirected and monochromatic case). The characterization is described in the next section. It is more involved than the case of 2-vertex multigraphs, but this should not be surprising as ternary structures tend to be substantially more difficult than their binary counterparts. An analogue in CSP is the dichotomy of binary CSP proved by Schaefer in the 70's [20] followed by the characterization of CSP into ternary structures by Bulatov almost 30 years later [5].

2 Preliminaries and statement of our results

For the sake of brevity we reserve the term "graph" for a multigraph. We denote the set of vertices of a graph G by $V(G)$ and the set of edges by $E(G)$. For two vertices u, v of G we denote the number of distinct edges between u and v by

$m_G(u,v)$ and we say that uv is an $m_G(u,v)$-edge. The degree of vertex v of G is denoted by $\deg_G(v)$ (recall that in multigraphs, the degree of a vertex v is defined as the number of edges going to other vertices plus twice the number of loops at v, i.e. $deg_G(v) = 2m_G(v,v) + \sum_{u \neq v} m_G(u,v)$). By $N_G(v)$ we denote the multiset of neighbors of vertex v in G where the multiplicity of v in $N_G(v)$ is $2m_G(v,v)$ and for every $u \neq v$ the multiplicity is $m_G(u,v)$. We omit G in the subscript if G is clear from the context.

Suppose A and B are two multisets. Let A', resp. B' be the set of different elements from A, resp. B. We say that a mapping $g\colon A' \to B'$ is a bijection from A to B if for every $b' \in B'$ the sum of multiplicities of all elements from $g^{-1}(b')$ in A equals the multiplicity of b' in B (note that g is not necessarily a bijection between sets A' and B'). If C' is a set then by $A \cap C'$ we mean a multiset that contains only elements from $A' \cap C'$ with the multiplicities corresponding to A. We denote the sum of multiplicities of all elements in A by $|A|$.

Let G and H be graphs. A *homomorphism* $f : V(G) \to V(H)$ is an edge preserving mapping from $V(G)$ to $V(H)$. A homomorphism f is a *covering projection* if $N_G(v)$ is mapped to $N_H(f(v))$ bijectively for every $v \in V(G)$ (here we consider the multiset bijection). Note that by the definition a covering projection is not necessarily surjective. The notion of a covering projection is also known as a *locally bijective homomorphism* or simply a *cover*. In this paper we denote a covering projection f from G to H by $f\colon G \to H$.

Strictly speaking, a covering projection (as the notion follows from topology) should be defined by a pair of mappings – one on the vertices and one on the edges of the graphs involved. But it was shown in [16] (using König's theorem and 2-factorization of $2k$-regular graphs) that every cover (defined as above) can be extended to a topological covering projection $f : V(G) \cup E(G) \to V(H) \cup E(H)$.

In this paper we consider the following decision problem.

Problem: H-Cover
Parameter: Fixed graph H.
Input: Graph G.
Task: Does there exist a covering projection $f : G \to H$?

Note that the problem H-Cover belongs to NP as we can guess a mapping $f\colon V(G) \to V(H)$ and verify if f is a covering projection in polynomial time. This means that in our NP-completeness results we only prove the NP-hardness part.

An equitable partition of a graph G is a partition of its vertex set into blocks $B_1, \dots, B_d$ such that for every $i, j = 1, \dots, d$ and every vertex v in B_i it holds that $|N_G(v) \cap B_j| = r_{i,j}$ (recall that $N_G(v)$ is generally a multiset). We call the matrix $M = (r_{i,j})$ corresponding to the coarsest equitable partition $B_1, \dots, B_d$ of G (ordered in some canonical way; see Corneil and Gotlieb [6]) the degree refinement matrix of G, denoted by $drm(G)$, and we say that G is a d-block graph. Note that 1-block graphs are exactly regular graphs (despite the fact that vertices can contain a different number of loops).

It is also known that if G covers H via a covering f, then $drm(G) = drm(H)$. In particular, f preserves the coarsest equitable partition of G, i.e., if $B'_1, \dots, B'_d$, resp. $B_1, \ldots, B_d$ are the blocks in the partition of G, resp. H then $f(B'_i) = B_i$ for every $i = 1, \ldots, d$. Since the matrix $drm(G)$ can be computed in time polynomial in the size of G, in this paper, we assume that $drm(G) = drm(H)$.

For every quadruplet of non-negative integers k, l, x, y we define a graph $S(k, l, x, y)$ on the vertex set $\{a, b, c\}$ with the following edge multiplicities (see Figure 1):

- $m(a, c) = m(b, c) = k$
- $m(c, c) = l$
- $m(a, a) = m(b, b) = x$
- $m(a, b) = y$

In this paper we focus on graphs H having exactly three vertices. For such graphs we give the full computational complexity characterization of H-Cover. More precisely, we show the following P/NP-completeness dichotomy.

Observation 1 *Let H be a 3-block graph on three vertices. Then H-*Cover *is polynomially solvable.*

Theorem 1. *Let H be a 2-block graph on three vertices. If H is isomorphic to $S(k', l, x, 0), S(k', l, 0, y)$ or $S(2, l, 0, 0)$, where $k' \in \{0, 1\}$ and $l, x, y \geq 0$, then H-*Cover *is polynomially solvable. Otherwise H-*Cover *is NP-complete.*

Theorem 2. *Let H be a t-regular graph on three vertices. If H is disconnected or $t \leq 2$, then H-*Cover *is polynomially solvable. Otherwise, H-*Cover *is NP-complete.*

Note that whenever H-Cover is polynomially solvable then we are able to find a corresponding covering projection in polynomial time, as well. That follows directly from the proofs of Observation 1, Theorem 1, and Theorem 2.

Observation 1 follows from the fact that if $drm(G) = drm(H)$ then the only mapping $f: V(G) \to V(H)$ that preserves the blocks is a covering projection.

In Section 3 we state the necessary lemmata for the proof of Theorem 1. Section 4 is devoted to the proof of Theorem 2. All polynomial cases are covered by Lemma 5. We then introduce a new decision problem - H-Cover*. We prove that this problem is NP-complete for all connected t-regular graphs H with $t \geq 4$. The proof is based on mathematical induction where we are able to use a stronger induction hypothesis than with simple H-Cover. NP-hardness of H-Cover then follows from the fact that H-Cover* is reducible to H-Cover in polynomial time. Note that due to space limitation only the full version of the paper will contain all necessary lemmata and proofs.

Let us give a few more technical definitions and notations. Throughout the rest of the paper we reserve the letter H for a graph on 3 vertices a, b, and c.

Let m, n, z be integers such that $m \geq n > 0$ and $z \geq 0$. We define a graph $H(m, n, z)$ to be the graph on the vertex set $\{a, b, c\}$ such that (see Figure 1):

- $m(a, a) = m$
- $m(b, b) = n$
- $m(a, b) = z$
- $m(b, c) = z + 2m$
- $m(a, c) = z + 2n$
- $m(c, c) = 0$

Let G, F and H be graphs. From the definition of a covering projection it is easy to show that if $f: G \to F$ and $g: F \to H$ are two covering projections then the composition $g \circ f: G \to H$ is also a covering projection. Since every graph

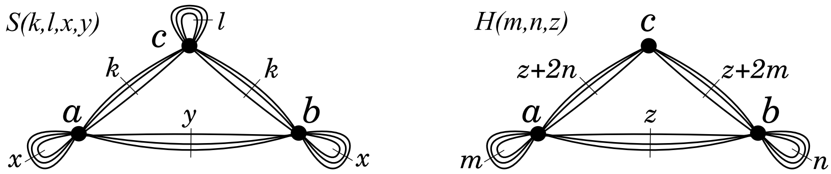

Fig. 1. The graphs $S(k,l,x,y)$ and $H(m,n,z)$.

isomorphism is a covering projection, every time we investigate the complexity of H-Cover where H is isomorphic to $S(k,l,x,y)$ or $H(m,n,z)$, we can and we will assume that $H = S(k,l,x,y)$ or $H = H(m,n,z)$.

By a boundary $\delta_G(F)$ of an induced subgraph F of a graph G we mean the subset of vertices of F that are adjacent to at least one vertex outside F.

Let A, B be sets and let $f\colon A \to B$ be a mapping. Then we define $f(A) = \bigcup_{a\in A}\{f(a)\}$. If $f(A)$ contains only one element, say x, then we simply write $f(A) = x$ instead of $f(A) = \{x\}$.

3 Complexity for 2-block graphs on three vertices

In this section we provide the proof of Theorem 1. We will assume that H is a 2-block graph with the blocks $\{a,b\}$ and $\{c\}$. From the definition of an equitable partition we have $deg_H(a) = deg_H(b) \neq deg_H(c)$. The next proposition shows the connection between graphs $S(k,l,x,y)$ and 2-block graphs.

Proposition 1. *Every 2-block graph H on three vertices is isomorphic to some $S(k,l,x,y)$, where $2x + y \neq 2l + k$.*

Proof. Since we cannot distinguish vertices a and b in the block $\{a,b\}$ we have $m(a,a) = m(b,b) = x$ and $m(a,c) = m(b,c) = k$. This means that H is isomorphic to $S(k,l,x,y)$, where $l = m(c,c)$ and $y = m(a,b)$. The inequality $2x + y \neq 2l + k$ then follows directly from the fact that $deg_H(a) \neq deg_H(c)$. □

Before we proceed to the proof of Theorem 1 we split all 2-block graphs into three subsets and show the complexity separately for each subset. Figure 2 shows how we split these graphs, and shows also the computational complexity of H-Cover for the graphs H in the corresponding subset.

Lemma 1. *Let H be a 2-block graph on three vertices. If H is isomorphic to $S(k',l,x,0)$, $S(k',l,0,y)$ or $S(2,l,0,0)$ for some $k' \in \{0,1\}$ and $l,x,y \geq 0$ then H-Cover is polynomially solvable.*

Proof. Let G be the input to H-Cover and let AB, resp. C be the block of G that corresponds to the block $\{a,b\}$, resp. $\{c\}$ of H.

First suppose that H is isomorphic to $S(k',l,x,0)$ or $S(k',l,0,y)$. We will construct a conjunctive normal form boolean formula φ_G with clauses of size 2, such that φ_G is satisfiable if and only if G covers H.

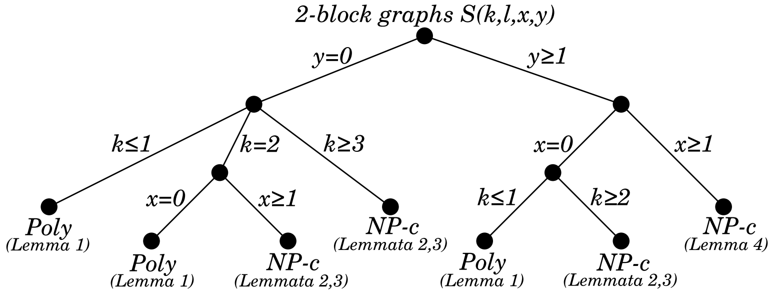

Fig. 2. The partition of 2-block graphs. Leaf vertices denote the computational complexity of H-Cover for the corresponding graph H.

Let the variables of φ_G be $\{x_u | u \in AB\}$ and for each $u, v \in AB$ we add to φ_G the following clauses:

- $(x_u \vee x_v)$ and $(\neg x_u \vee \neg x_v)$, if $u \neq v$ and u, v share a neighbor in C
- $(x_u \vee \neg x_v)$ and $(\neg x_u \vee x_v)$, if $uv \in E(G)$ and $H = S(k', l, x, 0)$
- $(x_u \vee x_v)$ and $(\neg x_u \vee \neg x_v)$, if $uv \in E(G)$ and $H = S(k', l, 0, y)$

Suppose that φ_G is satisfiable and fix one satisfying evaluation of variables. Define a mapping $f\colon V(G) \to V(H)$ by:

- $f(u) = a$, if $u \in AB$ and x_u is positive
- $f(u) = b$, if $u \in AB$ and x_u is negative
- $f(u) = c$, if $u \in C$

It is a routine check to show that f is a covering projection from G to H. On the other hand, if $f\colon G \to H$ is a covering projection then we can define an evaluation of φ_G such that x_u is positive if and only if $f(u) = a$. Such an evaluation satisfies the formula φ_G since there is exactly one positive literal in every clause. The fact that the size of φ_G is polynomial in the size of G and 2-SAT is polynomially solvable implies that H-Cover is polynomially solvable.

In the rest of the proof we suppose that $H = S(2, l, 0, 0)$. In this case the graph G covers H if and only if we can color the vertices of AB by two colors, say black and white, in such a way that for each $u \in C$ exactly two out of four vertices from $N_G(u) \cap AB$ are black.

We construct an auxiliary 4-regular graph G'. Let $V(G') = C$ and let edges of G' correspond to the vertices of AB, and connect its two neighbors in C. Note that G' can generally contain loops and multi-edges.

Then the coloring of vertices of AB in G corresponds to the coloring of edges of G' such that the black edges induce a 2-factor of G'. The problem of deciding the existence of a 2-factor in a 4-regular graph can be solved in polynomial time. In fact, such a 2-factor always exists and can be find in polynomial time. $\square$

In Lemma 2 we deduce NP-hardness of H-Cover from the following problem.

Problem: m-in-$2m$-SAT$_q$
Input: A formula φ in CNF where every clause contains exactly $2m$ variables

without negation and every variable occurs in φ exactly q times.
Task: Does there exist an evaluation of the variables of φ such that every clause contains exactly m positively valued variables?

Kratochvíl [14, Corollary 1] shows that this problem is NP-complete for every $q \geq 3$ and $m \geq 2$. If formula φ is a positive instance of m-in-$2m$-SAT$_q$ we simply say that φ is m-in-$2m$ *satisfiable*.

For the purposes of our NP-hardness deductions in Lemma 2 we will build a specific gadget according to the following needs:

Definition 1 (Variable gadget). *Let $H = S(k, l, x, y)$ and let F be a graph with $2q$ specified vertices $S = \{s_1, \ldots, s_q\}$ and $S' = \{s'_1, \ldots, s'_q\}$ of degree one. Let V, resp. V' be the set of neighbors of vertices in S, resp. S' in F. Suppose that whenever F is an induced subgraph of G with $\delta_G(F) \subseteq S \cup S'$ and $f : G \to H$ is a covering projection then $f(S \cup S') = c$ and one of the following occurs:*

i) $f(V) = a$ and $f(V') = b$	*iii) $f(V \cup V') = a$*
ii) $f(V) = b$ and $f(V') = a$	*iv) $f(V \cup V') = b$*

Furthermore, suppose that any mapping $f : S \cup S' \cup V \cup V' \to V(H)$ such that $f(S \cup S') = c$ and satisfying i) or ii) can be extended to $V(F)$ in such a way that for each $u \in V(F) \setminus (S \cup S')$ the restriction of f to $N_F(u)$ is a bijection to $N_H(f(u))$.

We denote such F by $VG_H(q)$ and we call it a variable gadget *of size q.*

The next lemma shows how we use variable gadgets while Lemma 3 proves that $VG_H(q)$ exists for some graphs $S(k, l, x, 0)$, $S(2, l, x, 0)$, and $S(k, l, 0, y)$. Note that in Definition 1 and Lemma 2 we do not use the fact that H is a 2-block graph. Hence, we can use this lemma also in Section 4.

Lemma 2. *Let $k \geq 2$ and let $H = S(k, l, x, y)$. If for some $q \geq 3$ there exists a variable gadget $VG_H(q)$ then H-COVER is NP-complete.*

Proof. We deduce NP-hardness of H-COVER from k-IN-$2k$-SAT$_q$. Let φ be an instance of k-IN-$2k$-SAT$_q$. Let $x_1, x_2, \ldots, x_n$, resp. $C_1, C_2, \ldots, C_m$ be the variables, resp. clauses of φ. For every clause C_i denote the variables in C_i by $l_i^1, \ldots, l_i^{2k}$ (recall that all variables have only positive appearances in φ). We construct a graph G_φ such that G_φ covers H if and only if φ is k-in-$2k$ satisfiable.

We start the construction of G_φ by taking vertices $c_1, \ldots, c_m, c'_1, \ldots, c'_m$ (corresponding to the clauses of φ) and we add l loops to each of them. For every variable x_i we take a copy $VG^i(q)$ of variable gadget $VG_H(q)$. Denote the copy of S, S', V, resp. V' in $VG^i(q)$ simply by S^i, S'^i, V^i, reps. V'^i. For every occurrence of x_i in C_j we identify one vertex from S^i, resp. S'^i with c_j, resp. c'_j. We do it in such a way that every vertex from $S^i \cup S'^i$ is identified exactly once, see Figure 3.

We claim that G_φ covers H if and only if φ is k-in-$2k$ satisfiable.

Suppose that there exists a covering projection $f : G_\varphi \to H$. We define an evaluation of the variables of φ such that x_i is true if and only if $f(V^i) = a$.

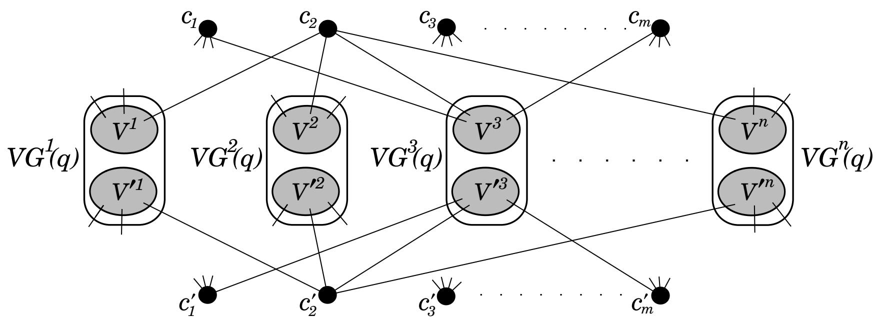

Fig. 3. The construction of the graph G_φ for $k = 2$ and $q = 3$. In this example φ contains a clause $C_2 = (x_1 \wedge x_2 \wedge x_3 \wedge x_n)$ and a variable x_3 appears in clauses C_1, C_2 and C_m.

From the properties of a variable gadget we know that $f(c_j) = c$ for every $j = 1, \ldots, m$. Then $|N_{G_\varphi}(c_j) \cap f^{-1}(a)| = |N_{G_\varphi}(c_j) \cap f^{-1}(b)| = k$. This means that in every clause of φ there is exactly k positive as well as negative variables.

For the opposite implication we fix one satisfying evaluation of φ. We define a mapping $f\colon V(G_\varphi) \to V(H)$ in the following way:

- $f(c_j) = f(c'_j) = c$, for all $j = 1, \ldots, m$
- $f(V^i) = a$ and $f(V'^i) = b$, if x_i is a positive variable
- $f(V^i) = b$ and $f(V'^i) = a$, if x_i is a negative variable

Then for each $i = 1, \ldots, n : f(S^i) = c$ and $f(V^i) \neq f(V'^i)$. By the definition of a variable gadget we know that f can be extended to every $VG^i(q)$ in such a way that for each $u \in V(VG^i(q)) \setminus (S^i \cup S'^i)$: the restriction of f to $N_{G_\varphi}(u)$ is a bijection to $N_H(f(u))$. It is a routine check to show that such a mapping f is a covering projection from G_φ to H. $\square$

Lemma 3. *If a* 2*-block graph H is one of the following:*

a) $S(k, l, x, 0)$*, where* $k \geq 3$*,* $l \geq 0$ *and* $x \geq 0$
b) $S(2, l, x, 0)$*, where* $l \geq 0$ *and* $x \geq 1$
c) $S(k, l, 0, y)$*, where* $k \geq 2$*,* $l \geq 0$ *and* $y \geq 1$

then there exists a variable gadget $VG_H(q)$ *for some* $q \geq 3$.

Proof. Depending on which of $a), b)$ and $c)$ holds for the graph H, we define $VG_H(q)$ and the corresponding sets S and S' as depicted in Figure 4. Note that in the case $a), b)$, resp. $c)$ we have that q is equal to $k, 4$, resp. $2k$.

The fact that the depicted graphs are really variable gadgets follows from a case analysis. Figure 4 also shows how one particular mapping $f\colon S \cup S' \cup V \cup V' \to H$ (where V, resp. V' are the neighbors of S, resp. S') can be extended to all vertices of $VG_H(q)$. Other conditions from the definition of $VG_H(q)$ follow from the fact that H has two blocks. $\square$

Lemma 4. *Let* $H = S(k, l, x, y)$ *be a* 2*-block graph where* $k, l \geq 0$ *and* $x, y \geq 1$*. Then* H-Cover *is NP-complete.*

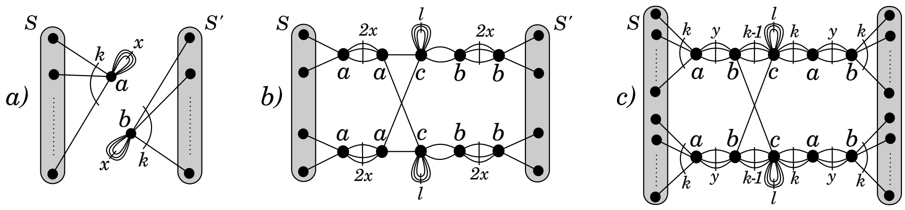

Fig. 4. Examples of the variable gadgets for the cases $a), b)$ and $c)$.

Proof. Kratochvíl et al. [16, Theorem 11] proved that if H' is a graph on two vertices L and R such that $x = m_{H'}(L, L) = m_{H'}(R, R) \geq 1$ and $y = m_{H'}(L, R) \geq 1$, then H'-Cover is NP-complete.

We deduce NP-hardness of H-Cover from H'-Cover. Let G' be an instance of H'-Cover. We construct a graph G such that G covers H if and only if G' covers H'.

We start the construction of G by taking two copies G^1 and G^2 of G'. Denote the copy of vertex $v \in V(G')$ in G^1, resp. G^2 by v^1, resp. v^2. For every $v \in V(G')$ we add to G a new vertex u_v with l loops and k-edges v^1u_v and v^2u_v.

Suppose that $f\colon G \to H$ is a covering projection. Then $f(u_v) = c$ for every $v \in V(G')$ and f restricted to G^1 is a covering projection to H'. This means that G' covers H'.

For the opposite implication suppose that $f'\colon G' \to H'$ is a covering projection. We define a mapping $f\colon V(G) \to V(H)$ in the following way:

- $f(u_v) = c$
- $f(v^1) = f'(v)$
- $f(v^2) = a$ if $f(v^1) = b$, and $f(v^2) = b$ otherwise

for every $v \in V(G')$. It is a routine check to show that f is a covering projection from G to H. □

Next we proceed to the proof of Theorem 1.

Proof (of Theorem 1). The polynomial cases are settled by Lemma 1. The cases where $x, y \geq 1$ follow from Lemma 4. All other cases follow from Lemmata 2 and 3 (see Figure 2). □

4 Complexity for 1-block graphs on three vertices

In this section we focus on 1-block graphs H, i.e. regular graphs. We provide several definitions and lemmata that help us prove Theorem 2. The next lemma settles the polynomial cases.

Lemma 5. *Let H be a t-regular graph on three vertices. If H is disconnected or $t \leq 2$, then H-Cover is polynomially solvable.*

Proof. Let G be a t-regular graph. Let us first suppose that H is disconnected. Without loss of generality suppose that $m_H(a,c) = m_H(b,c) = 0$. We define a mapping $f\colon V(G) \to V(H)$ by $f(u) = c$ for every $u \in V(G)$. Then mapping f is a covering projection from G to H by the definition.

If H is connected and $t \leq 2$, then $t = 2$ and H is a triangle. A 2-regular graph G covers the triangle if and only if G consists of disjoint cycles of lengths divisible by 3. This condition can be easily verified in linear time. □

For the NP-hardness part of Theorem 2 we use a reduction from a problem we call H-Cover*. To define H-Cover* we need the following definitions.

Definition 2. *Let G be a graph on $3n$ vertices and let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a partition of its vertices into n sets of size 3. Then we say that $\mathcal{A}$, resp. pair $(G, \mathcal{A})$ is a* 3-partition, *resp.* graph 3-partition. *Moreover, if $f\colon V(G) \to \{a,b,c\}$ is a mapping such that $f(A_i) = \{a,b,c\}$ for every $A_i \in \mathcal{A}$ then we say that f* respects *the 3-partition $\mathcal{A}$.*

Definition 3. *We say that a graph 3-partition $(G, \mathcal{A})$ covers* graph H if there exists a covering projection $f^*\colon G \to H$ that respects the 3-partition $\mathcal{A}$. We denote such a mapping by "$\to^*$" and call it a* covering projection* *or simply a* cover*.

Definition 4. *Let $(G, \mathcal{A})$ be a graph 3-partition and let H be a graph. If the existence of a covering projection $f\colon G \to H$ implies the existence of a covering projection* $f^*\colon (G, \mathcal{A}) \to^* H$, then we say that $(G, \mathcal{A})$ is* nice *for H.*

Note it follows from these definitions that if G does not cover H then any graph 3-partition $(G, \mathcal{A})$ is nice for H.

Problem: H-Cover*
Parameter: Fixed graph H.
Input: Nice graph 3-partition $(G, \mathcal{A})$ for H.
Task: Does there exist a covering projection* $f\colon (G, \mathcal{A}) \to^* H$?

Similarly as H-Cover also the H-Cover* problem belongs to the class NP. This means that to show NP-completeness of H-Cover* we only need to prove NP-hardness.

Observation 2 *Let H be a graph. Then H-Cover* is polynomially reducible to H-Cover.*

Proof. Suppose that $(G, \mathcal{A})$ is an instance of H-Cover*. Since $(G, \mathcal{A})$ is nice for H we know that $(G, \mathcal{A})$ covers* H if and only if G covers H, which concludes the proof. □

This observation allows us to prove NP-hardness of H-Cover* instead of H-Cover. We do this by mathematical induction. The key advantage of H-Cover* is that we can use a stronger induction hypothesis.

Theorem 3. *Let H be a connected t-regular graph on three vertices and $t \geq 4$. Then H-*Cover** *is NP-complete.*

In the rest of the paper we prove Theorem 3. We assume that H is a connected t-regular graph and $t \geq 4$.

The following lemma deduces NP-hardness of H-Cover* for a very special graph H, and will serve as an illustration of such deductions. NP-hardness of H-Cover* is deduced from a 3-edge coloring problem. Holyer [13] proved that this problem is NP-complete even for simple cubic graphs. Denote the 3-edge coloring problem for cubic graphs by 3-ECol.

Lemma 6. *Let $H = S(1,1,1,1)$. Then H-*Cover** *is NP-complete.*

Proof. We reduce the NP-hard problem 3-ECol to H-Cover*. For every simple cubic graph F we construct a graph 3-partition $(G_F, \mathcal{A})$ such that $(G_F, \mathcal{A})$ covers* H if and only if F is 3-edge colorable.

For every vertex $u \in V(F)$ we insert to G_F vertices u_1, u_2, u_3 and we add 1-edges u_1u_2, u_2u_3 and u_3u_1. For every edge $uv \in E(F)$ we choose vertices u_i and v_j and we add 2-edge u_iv_j to G_F. We choose indices the i and j in such a way that the final graph G_F is 4-regular. We define the 3-partition $\mathcal{A}$ as $\bigcup_{u \in V(F)} \{\{u_1, u_2, u_3\}\}$.

We prove that $(G_F, \mathcal{A})$ is nice for H. Let $f : G_F \to H$ be a covering projection. Clearly all 2-edges of G_F must be mapped by f to loops of H. This implies that for every $u \in V(F)$ is $f(u_1, u_2, u_3) = \{a, b, c\}$ and so f respects $\mathcal{A}$.

Suppose that $f^* \colon (G_F, \mathcal{A}) \to^* H$ is a covering projection*. We know that every 2-edge u_iv_j corresponds to an edge uv of F and $f^*(u_i) = f^*(v_j)$. We define a coloring $col \colon E(F) \to V(H)$ by $c(uv) = f^*(u_i)$. The fact that f respects the 3-partition $\mathcal{A}$ implies that col is a proper 3-edge coloring of F.

In the rest of the proof suppose that $col \colon E(F) \to V(H)$ is a proper 3-edge coloring of F. We show that there exists a covering projection* $f^* \colon (G_F, \mathcal{A}) \to^* H$. For every 2-edge u_iv_j of G_F we define $f^*(u_i) = f^*(v_j) = col(uv)$. Since col is a proper 3-edge coloring of F we have $\{f^*(u_1), f^*(u_2), f^*(u_3)\} = \{a, b, c\}$ for every $u \in V(F)$. This means that f^* respects the 3-partition $\mathcal{A}$. It is a routine check to show that f^* is a covering and consequently a covering projection*. □

As already mentioned, due to space limitation we have in this extended abstract removed the remainder of the lemmata needed for the proof of Theorem 3. These can be found in the full version of the paper. We proceed to the proof of Theorem 2 that handles the complexity of H-Cover for all 1-block graphs H on three vertices.

Proof (of Theorem 2). Lemma 5 covers all polynomial cases while Theorem 3 with Observation 2 covers the NP-complete cases. □

5 Conclusion

We have settled the computational complexity of H-cover for all multigraphs on three vertices. Not surprisingly, the characterization is substantially more involved than the characterization of the 2-vertex case. These results constitute an important step towards the goal of a full dichotomy for complexity of H-COVER of simple graphs, a goal that requires a full dichotomy also for colored mixed multigraphs, as shown in [16], and in particular a dichotomy for the multigraphs handled in this paper.

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