Between Treewidth and Clique-width

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Abstract

Many hard graph problems can be solved efficiently when restricted to graphs of bounded treewidth, and more generally to graphs of bounded clique-width. But there is a price to be paid for this generality, exemplified by the four problems MAXCUT, GRAPH COLORING, HAMILTONIAN CYCLE and EDGE DOMINATING SET that are all FPT parameterized by treewidth but none of which can be FPT parameterized by clique-width unless FPT = W[1], as shown by Fomin et al [7, 8][1]. We therefore seek a structural graph parameter that shares some of the generality of clique-width without paying this price.

Based on splits, branch decompositions and the work of Vatshelle [18] on Maximum Matching-width, we consider the graph parameter sm-width which lies between treewidth and clique-width. Some graph classes of unbounded treewidth, like distance-hereditary graphs, have bounded sm-width. We show that MAXCUT, GRAPH COLORING, HAMILTONIAN CYCLE and EDGE DOMINATING SET are all FPT parameterized by sm-width.

1 Introduction

Many hard problems can be solved efficiently when restricted to graphs of bounded treewidth or even graphs of bounded clique-width. A celebrated algorithmic metatheorem of Courcelle [5] states that any problem expressible in monadic second-order logic (MSO$_2$) is fixed parameter tractable (FPT) when parameterized by the treewidth of the input graph. This includes many problems like DOMINATING SET, GRAPH COLORING, and HAMILTONIAN CYCLE. Likewise, Courcelle et al [4] show that the subset of MSO$_2$ problems expressible in MSO$_1$-logic, which does not allow quantification over edge sets, is FPT parameterized

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[1] In [8] the problems GRAPH COLORING, HAMILTONIAN CYCLE and EDGE DOMINATING SET are shown to be FPT only if FPT = W[1], while in [7] the authors focus on showing that neither of MAXCUT, HAMILTONIAN CYCLE and EDGE DOMINATING SET can have a $f(k)n^{o(k)}$ algorithm parameterized by clique-width unless the Exponential Time Hypothethis fails. However, in [7] they also show that MAXCUT is FPT only if FPT = W[1].

by clique-width. Originally this required a clique-width expression as part of the input, but this restriction was removed when Oum and Seymour [15] gave an algorithm that, in time FPT parameterized by the clique-width k of the input graph, finds a $2^{O(k)}$-approximation of an optimal clique-width expression.

Clique-width is stronger than treewidth, in the sense that bounded treewidth implies bounded clique-width [3] but not vice-versa, as exemplified by the cliques. Can we hope to find a graph width parameter lying between treewidth and clique-width for which all MSO$_2$ problems are FPT? Alas no, under the minimal requirement that cliques should have bounded width, Courcelle et al [4] showed that this would imply P=NP for unary languages. There are some basic problems belonging to MSO$_2$ but not MSO$_1$, like MaxCut, Graph Coloring, Hamiltonian Cycle and Edge Dominating Set. Fomin et al [7, 8] showed that none of these four problems can be FPT parameterized by clique-width, unless FPT = W[1]. Can we find a graph width parameter lying between treewidth and clique-width for which at least these four problems are FPT? Note that one can define trivial parameters having these properties (e.g. value equal to clique-width if this is at most 3, and otherwise equal to treewidth) but can we find one yielding new FPT algorithms for certain natural graph classes? This is the question motivating the present paper, and the answer is yes. We give a parameter which is low when the graph has low treewidth in local parts, and where each of these parts are connected together in a dense manner.

Before explaining our results, let us mention some related work. A class of graphs can have bounded treewidth only if it is sparse. Indeed, the introduction of clique-width was motivated by the desire to extend algorithmic results for bounded treewidth also to some dense graph classes. Let us say that a parameter x is weaker than parameter y, and y stronger than x, if for any graph class, a bound on x implies a bound on y. Alternatively, x and y are of the same strength, or incomparable. Thus, clique-width is stronger than treewidth. As we discussed above there are limitations inherent in clique-width and there have been several suggestions for width parameters weaker than clique-width but still bounded on some dense graph classes. In particular, let us mention four parameters: neighborhood diversity introduced by Lampis in 2010 [13], twin-cover introduced by Ganian in 2011 [10], shrub-depth introduced by Ganian et al in 2012 [11], and modular-width proposed by Gajarský et al in 2013 [9]. All these parameters are bounded on some dense classes of graphs, all of them are weaker than clique-width, but none of them are stronger than treewidth. Modular-width is stronger than both neighborhood diversity and twin-cover, but incomparable to shrub-depth [9]. Graph Coloring and Hamiltonian Cycle are W-hard parameterized by shrub-depth but FPT parameterized by modular-width, as recently shown by Gajarský et al [9] which also leaves as an open problem the complexity of MaxCut and Edge Dominating Set parameterized by modular-width.

In our quest for a parameter stronger than treewidth and weaker than clique-width, for which the four basic problems MaxCut, Graph Coloring, Hamiltonian Cycle and Edge Dominating Set become FPT, we are faced with two tasks when given a graph G with parameter-value k: we need an FPT

algorithm returning a decomposition of width $f(k)$, and we need a dynamic programming algorithm solving each of the four basic problems in FPT time when parameterized by the width of this decomposition. The requirement that the parameter be stronger than treewidth is a guarantee that it shares this property with clique-width and will capture large tree-like classes of graphs, also when some building blocks are dense. Arguably the most natural way to hierarchically decompose a graph are the so-called branch decompositions, originating in work of Robertson and Seymour [17] and used in the definition of both rank-width [15] and boolean-width [1], two parameters of the same strength as clique-width. Branch decompositions over the vertex set of a graph can be viewed as a recursive partition of the vertices into two parts, giving a rooted binary tree where each edge of the tree defines the cut given by the vertices in the subtree below the edge. Using any symmetric cut function defined on subsets of vertices we can define a graph width parameter as the minimum, over all branch decompositions, of the maximum cut-value over all edges of the branch decomposition tree. Recently, Vatshelle [18] gave a cut-function based on the size of a maximum matching, whose associated graph width parameter, called MM-width, has the same strength as treewidth.

In Section 2, based on the work of Vatshelle, we define the parameter split-matching-width, denoted sm-width, by a cut function based on maximum matching unless the cut is a split, i.e. a complete bipartite graph plus some isolated vertices. The sm-width parameter is stronger than treewidth and weaker than clique-width. It is also stronger than twin-cover but incomparable with neighborhood diversity, shrub-depth and modular-width. We finish Section 2 by showing that maximum matching is a submodular cut function. In Section 3 this is used together with an algorithm for split decompositions by Cunningham [6] and an algorithm for branch decompositions based on submodular cut functions by Oum and Seymour [15] to design an algorithm that given a graph G with sm-width k computes a branch decomposition of sm-width $O(k^2)$, in time $O^*(8^k)$. To our knowledge the use of split decompositions to compute a width parameter is novel.

In Section 4, using a slightly non-standard framework for dynamic programming, we are then able to solve the four basic problems MaxCut, Graph Coloring, Hamiltonian Cycle and Edge Dominating Set, by runtimes $\mathcal{O}^*(8^k), \mathcal{O}^*(k^{5k}), \mathcal{O}^*(2^{24k^2})$, and $\mathcal{O}^*(3^{5k})$ respectively, when given a branch decomposition of sm-width k. In Section 5 we show that some well-known graph classes of bounded clique-width also have bounded sm-width, e.g. distance-hereditary graphs have clique-width at most three and sm-width one. We also show that a graph whose twin-cover value is k will have sm-width at most k, and discuss classes of graphs where our results imply new FPT algorithms. In Section 6 we give a short summary of our results and end the paper by some concluding remarks.

2 Preliminaries

We deal with finite, simple, undirected graphs $G = (V, E)$ and denote also the vertex set by $V(G)$ and the edge set by $E(G)$. For the subgraph of G induced by $S \subseteq V(G)$ we write $G[S]$, and for disjoint sets $A, B \subseteq V(G)$ we denote the induced bipartite subgraph having vertex set $A \cup B$ and edge set $\{uv : u \in A, v \in B\}$ as $G[A, B]$. For a set E' of edges, we denote its endpoints by $V(E')$. For two graphs G_1 and G_2, we denote by $G_1 + G_2$ the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_1)$. If a set of vertices V' or a set of edges E' is written where a graph is expected (e.g., $G_1 + E'$ or $G_1 + V'$), we interpret E' as the graph $(V(E'), E')$ and V' as the graph $(V', \emptyset)$. For $v \in V(G)$ we write $N(v)$ or $N_G(v)$ for the neighbors of v and for $S \subseteq V(G)$ we denote the neighborhood of S by $N(S) = \bigcup_{a \in S} N(a) \setminus S$ or $N_G(S)$; note that $N(S) \cap S = \emptyset$. A matching is a set of edges having no endpoints in common.

A *split* of a connected graph G is a partition of $V(G)$ into two sets V_1, V_2 such that $|V_1| \geq 2$, $|V_2| \geq 2$ and every vertex in V_1 with a neighbor in V_2 has the same neighborhood in V_2 (this also means every vertex in V_2 with a neighbour in V_1 has the same neighbourhood in V_1). A graph G with a split (V_1, V_2) can be *decomposed* into a graph G_1 and a graph G_2 so that G_1 and G_2 is the induced subgraph of G on V_1 and V_2, respectively, except that an extra vertex v, called a *marker*, is added, and also some extra edges are added to G_1 and G_2, so that $N_{G_1}(v) = N_G(V_2)$ and $N_{G_2}(v) = N_G(V_1)$. If a graph G can be decomposed to the two graphs G_1 and G_2, then G_1 and G_2 *compose* G. We denote this by $G = G_1 * G_2$. A graph that cannot be decomposed (i.e., a graph without a split) is called a *prime*. As all graphs of at most three vertices trivially is a prime, when a prime graph has more than three vertices, it is called a *non-trivial* prime graph. A *split decomposition* of a graph G is a recursive decomposition of G so that all of the obtained graphs are prime. For a split decomposition of G into $G_1, G_2, \ldots, G_k$, a *split decomposition tree* is a tree T where each vertex corresponds to a prime graph and we have an edge between two vertices if and only if the prime graphs they correspond to share a marker. That is, the edge set of the tree is $E(T) = \{v_i v_j : v_i, v_j \in V(T) \text{ and } V(G_i) \cap V(G_j) \neq \emptyset\}$. To see that this is in fact a tree, we notice that T is connected and that we have an edge for each marker introduced. As there are exactly one less marker than there are prime graphs, T must be a tree. See Figure 1 for an example.

Given a split decomposition of graph G with prime graphs $G_1, G_2, \ldots, G_k$, we define $\text{tot}(v : G_i)$ recursively to be $\{v\}$ if $v \in V(G)$, and otherwise to be $\bigcup_{u \in V(G_j) \setminus \{v\}} \text{tot}(u : G_j)$ for the graph $G_j \neq G_i$ containing the marker v in the split decomposition. Another way of saying this latter part by the use of the split decomposition tree T is: if v is not in $V(G)$, then $\text{tot}(v : G_i)$ is defined to be the vertices of $V(G)$ residing in the prime graphs of the connected component in $T[V(T) - G_i]$ where v is also located. From this last definition, we observe that for a prime graph G_i in a split decomposition of G, the function tot on the vertices of G_i partitions the vertices of $V(G)$. For a set $V' \subseteq V(G_i)$, we define $\text{tot}(V' : G_i)$ to be the union of $\text{tot}(v : G_i)$ for all $v \in V'$. For a set $S \subseteq V(G)$, the inverse function $\text{tot}^{-1}(S : G_i)$, is defined as the minimal set

of vertices $V' \subseteq V(G_i)$ so that $S \subseteq \text{tot}(V' : G_i)$. We define the *active set* of a vertex $v \in G_i$, denoted $\text{act}(v : G_i)$ to be the vertices of $\text{tot}(v : G_i)$ that are contributing to the neighborhood of v in G_i. That is, $\text{act}(v : G_i)$ is defined as $N(V(G) \setminus \text{tot}(v : G_i))$. See Figure 1 for an example of tot() and act(). Note that if G has a split decomposition into prime graphs $G_1, \ldots, G_k$, then for any marker v there are exactly two prime graphs G_i and G_j containing v, and we have $\text{tot}(v : G_i) \cup \text{tot}(v : G_j) = V(G)$.
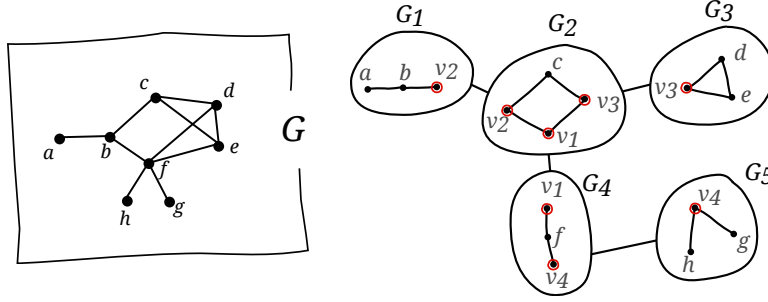

Figure 1: Split decomposition tree of a graph G. The markers of each prime graph are circled in red. An example of a split decomposition resulting in this tree is: $((G_1 * G_2) * G_3) * (G_4 * G_5)$. Note that $\text{tot}(\{v_1, v_3\} : G_2) = \{d, e, f, g, h\}$, $\text{act}(v_1 : G_4) = \{b, d, e\}$ and $\text{tot}^{-1}(a, f, g : G_2) = \{v_1, v_2\}$.

A *branch decomposition* (T, δ) of a graph G consists of a subcubic tree T (a tree of maximum degree 3) and a bijective function δ from the leaves of T to the vertices of G. For a graph G a *cut* $(A, \overline{A})$ for $A \subseteq V(G)$ is a bipartition of verices of G. For a cut (A, B) of G, we say the edges in G with one enpoint in A and the other in B *cross* the cut (A, B). In a branch decomposition $(T = (V_T, E_T), \delta)$ of a graph G, each edge $e \in E_T$ partitions $V(G)$ into two parts: the vertices mapped by δ from the leaves of one component of $T - e$, and the vertices mapped by δ from the leaves of the other component. Thus each edge of T induces a cut in G, namely the cut corresponding to that edge's bipartition of $V(G)$. For a graph G, a *cut function* $f : 2^{V(G)} \to \mathbb{N}$ is a symmetric ($f(A) = f(\overline{A})$) function on subsets of $V(G)$. For a branch decomposition (T, δ) of G its f-width, for a cut function f, is the maximum of $f(A)$ over all cuts $(A, \overline{A})$ of G induced by the edges of T. For a graph G, its f-width, for a cut function f, is the minimum f-width over all branch decompositions of G.

Vatshelle [18] defined the Maximum-Matching-width (MM-width) $\text{mmw}(G)$ of a graph G based on the cut function mm defined for any graph G and $A \subseteq V(G)$ by letting $\text{mm}(A)$ be the cardinality of a maximum matching of the bipartite graph $G[A, \overline{A}]$. In his work, Vatshelle shows that there is a linear dependency between the treewidth of a graph and the Maximum-Matching-width of the graph.

Theorem 1 ([18])**.** *Let G be a graph, then $\frac{1}{3}(\text{tw}(G)+1) \leq \text{mmw}(G) \leq \text{tw}(G)+1$*

In this paper we define the split-matching-width $\text{smw}(G)$ of a graph G based

on the cut function sm defined for any graph G and $A \subseteq V(G)$ by:

$$\mathrm{sm}(A) = \begin{cases} 1 \text{ if } \left(A, \overline{A}\right) \text{ is a split of } G \\ \mathrm{mm}(A) = \max\{|M| : M \text{ is a matching of } G[A, \overline{A}]\} \text{ otherwise} \end{cases}$$

A cut function $f : 2^{V(G)} \to \mathbb{N}$ is said to be submodular if for any $A, B \subseteq V(G)$ we have $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$. The following very general result of Oum and Seymour is central to the field of branch decompositions.

Theorem 2 ([15]). *For symmetric submodular cut-function f and graph G of optimal f-width k, a branch decomposition of f-width at most $3k + 1$ can be found in $\mathcal{O}^*(2^{3k+1})$ time.*

There is no abundance of submodular cut functions, but this result will be useful to us.

Theorem 3. *The cut function* mm *is submodular.*

Proof. Let G be a graph and $S \subseteq V(G)$. We will say that a matching $M \subseteq E(G)$ is a matching of S if each edge of M has exactly one endpoint in S, i.e. M is a matching of the bipartite graph $G[S, \overline{S}]$. To prove that mm is submodular, we will show that for any $A, B \subseteq V(G)$ and any matching $M_{A \cup B}$ of $A \cup B$ and $M_{A \cap B}$ of $A \cap B$, there must exist two matchings M_A of A and M_B of B so that the multiset of edges $M_A \uplus M_B$ is equal to the multiset $M_{A \cup B} \uplus M_{A \cap B}$. First notice that each edge of $M_{A \cup B}$ and $M_{A \cap B}$ is a matching of either A or B (or both). As the vertices in a matching have degree one, the multiset $M_{A \cup B} \uplus M_{A \cap B}$ of edges can be regarded as a set of vertex disjoint paths and cycles (note though, we might have cycles of size two, as the same edge might be in both of the matchings). We will show that for every such path or cycle P there exist matchings M_A^P for A and M_B^P for B so that $E(P) = E(M_A^P) \cup E(M_B^P)$. Note that this suffices to prove the statement, as there will then also exist matchings M_A of A and M_B of B so that $E(M_A) \uplus E(M_B) = E(M_{A \cup B}) \uplus E(M_{A \cap B})$, by taking M_A and M_B as the disjoint union of each of the smaller matchings, for A and B respectively, that exist for each path or cycle P in $M_{A \cup B} \uplus M_{A \cap B}$. Since these paths and cycles are vertex-disjoint M_A and M_B will be matchings.

Thus, let P be a path or a cycle from $M_{A \cup B} \uplus M_{A \cap B}$. If P only contains vertices of $A \cap B$ and $\overline{A \cup B}$, each edge of P is a matching of both A and B, so we have the matchings by setting $M_A = P \cap M_{A \cap B}$ and $M_B = P \cap M_{A \cup B}$. Since the edges of $E(P)$ alternate between $M_{A \cup B}$ and $M_{A \cap B}$, and since all edges from $M_{A \cup B}$ has an endpoint in $\overline{A \cup B}$ and all edges from $M_{A \cap B}$ has an endpoint in $A \cap B$, there can be at most one vertex v in P belonging to $(B \setminus A) \cup (A \setminus B)$ (it may help to look at Figure 2 where it is clear that no path alternating between blue and red edges can touch $(B \setminus A) \cup (A \setminus B)$ twice). If there exists such a vertex v, assume without loss of generality that $v \in B \setminus A$. As each edge in $M_{A \cap B} \cap P$ has exactly one endpoint in $A \cap B$, and P contains vertices only of $A \cap B, B \setminus A$ and $\overline{A \cup B}$, all the edges of $M_{A \cap B} \cap P$ has one endpoint in A and one endpoint in $(B \setminus A) \cup \overline{A \cup B} = \overline{A}$. So, $M_{A \cap B} \cap P$ is a matching of A. For

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$M_{A \cup B}$, by the same arguments, each edge in $M_{A \cup B} \cap P$ must have one endpoint in $\overline{A \cup B}$ and one endpoint in $(B \setminus A) \cup (A \cap B) = B$, making $M_{A \cup B} \cap P$ a matching of B. □
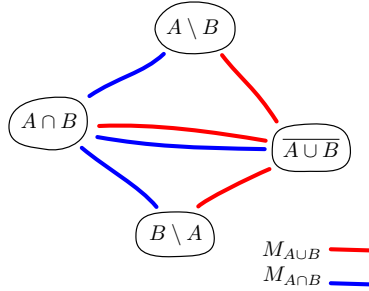

Figure 2: The edges of $M_{A \cap B}$ and $M_{A \cup B}$

3 Computing an approximate sm-width decomposition

In this section we design an algorithm that given a graph G finds a branch decomposition of G having sm-width $O(\text{smw}(G)^2)$, in time FPT parameterized by $\text{smw}(G)$. The algorithm has four main steps:

1. Find a split decomposition of G into prime graphs $G_1, G_2, ..., G_q$.

2. For each G_i find a branch decomposition (T_i, δ_i) of sm-width $O(\text{smw}(G_i))$.

3. For each G_i restructure (T_i, δ_i) into (T'_i, δ'_i) having the property that any cut of G_i, induced by an edge of (T'_i, δ'_i) and having split-matching value k, is lifted, by the split decomposition of G, to a cut of G having split-matching value $O(k^2)$.

4. Combine all the decompositions (T'_i, δ'_i) into a branch decomposition of G of sm-width $O(\text{smw}(G)^2)$.

For step 1 there exists a well-known polynomial-time algorithm by Cunningham [6] and even linear-time ones, see e.g. [2] and see also [16] for the use of split decompositions in general. For step 2 we are dealing with a prime graph G_i, which by definition has no non-trivial splits and hence $\text{sm}(V_i) = \text{mm}(V_i)$ for all $V_i \subseteq V(G_i)$ meaning that $\text{mmw}(G_i) = \text{smw}(G_i)$. Furthermore, by Theorem 3 the cut function defining mmw is submodular so we can apply the algorithm of Oum and Seymour from Theorem 2 to accomplish the task of step 2. Step 3 will require more work. Let us first give a sketch of step 4. Suppose for each prime graph G_i of a split decomposition of G we have calculated a branch decomposition (T'_i, δ'_i) for G_i. If for every cut $(X, V(G_i) \setminus X)$ of G_i induced by

an edge of (T'_i, δ'_i) we have $\text{sm}(\text{tot}(X : G_i)) \leq t$ for some value t, then we can generate a branch decomposition of G of sm-width at most t by for each pair of prime graphs G_i, G_j sharing a marker, identifying the two leaves of respectively (T'_i, δ'_i) and (T'_j, δ'_j) mapped to this marker (see figure 3).
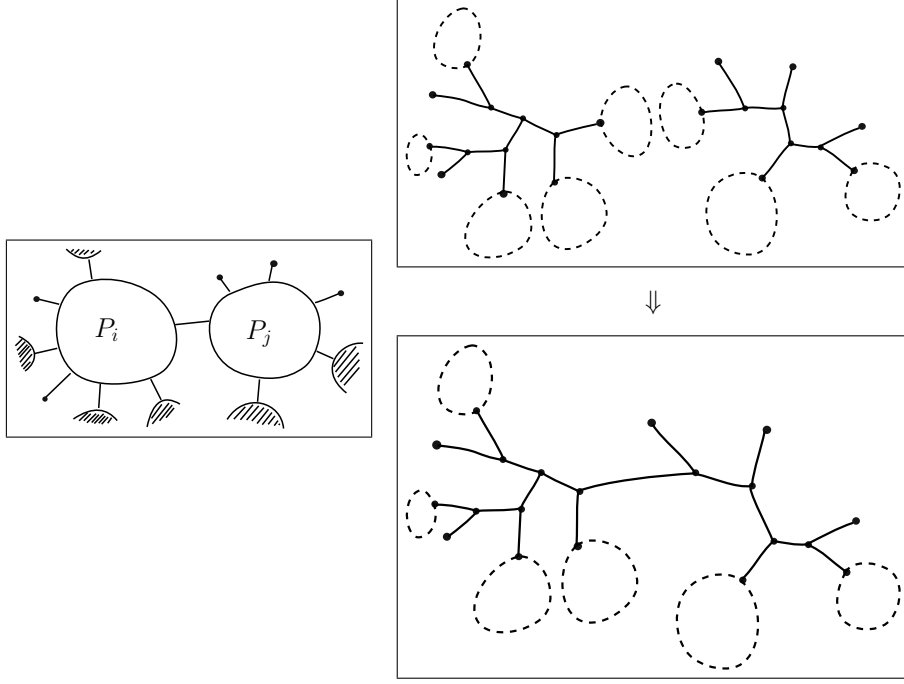

Figure 3: Combining the decomposition of prime graphs sharing a marker. The prime graphs in a split decomposition tree to the left and their branch decomposition trees – before and after combining them – to the right.

What remains is step 3, covered by Theorem 12. We need to relate $\text{sm}(A)$ of a cut $(A, V(G_i) \setminus A)$ in prime graph G_i, induced by an edge of (T'_i, δ'_i), to $\text{sm}(\text{tot}(A : G_i))$ of the associated cut $\left(\text{tot}(A : G_i), \overline{\text{tot}(A : G_i)} \right)$ in G. This we do by the series of lemmas from Lemma 4 to Lemma 10. The latter of these lemmas use the notion of a *heavy* pair of vertices: In a graph G with $smw(G) < k$ and split decomposition into prime graphs $G_1, G_2, ..., G_q$ we say that adjacent vertices $a, b \in V(G_i)$ are *heavy* if $|\text{act}(a : G_i)| \geq 3k$ and $|\text{act}(b : G_i)| \geq 3k$. The edge connecting a heavy pair is called a *heavy* edge.

Lemma 4. *Let G be a graph and P a non-trivial prime graph in a split decomposition of G. For any split (X, Y) of G, there exist a vertex $v \in V(P)$ such that either $X \subseteq \text{tot}(v : P)$ or $Y \subseteq \text{tot}(v : P)$.*

Proof. Let $X_P = \text{tot}^{-1}(X : P)$ and $Y_P = \text{tot}^{-1}(Y : P)$. Assume for contradiction that both $|X_P| \geq 2$ and $|Y_P| \geq 2$. Since $X \cup Y = V(G)$, we have

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$X_P \cup Y_P = V(P)$. The fact that $X_P \cup Y_P = V(P)$ and $|X_P| \geq 2$, $|Y_P| \geq 2$, $|V(P)| \geq 4$ implies that $V(P)$ has a partition into X'_P, Y'_P with $X'_P \subseteq X_P$ and $Y'_P \subseteq Y_P$, and $|X'_P| \geq 2$ and $|Y'_P| \geq 2$.

As $P[X'_P, Y'_P]$ is isomorphic to an induced subgraph of $G[X, Y]$, and (X, Y) is a split of G, (X'_P, Y'_P) must also be a split in P. However, as both X'_P and Y'_P have cardinality at least 2, this contradicts the fact that a prime graph does not have any splits. $\square$

Lemma 5. *Let G be a graph, P a non-trivial prime graph in a split decomposition of G and (T, δ) a branch decomposition of sm-width less than k. For $a \in V(P)$ and cut (X, Y) in (T, δ), if $|X \cap \mathrm{act}(a : P)| \geq k$ and $|Y \cap \mathrm{act}(N(a) : P)| \geq k$, then (X, Y) is a split.*

Proof. Since $X \cap \mathrm{act}(a : P)$ and $Y \cap \mathrm{act}(N(a) : P)$ form a complete bipartite graph with at least k vertices on each side, the mm-value of the cut (X, Y) must be at least k. And since (X, Y) is a cut in a branch decomposition of sm-width less than k, we conclude that (X, Y) must be a split. $\square$

Lemma 6. *For any two (not necessarily disjoint) vertex subsets A and B of $V(G)$, and in any branch decomposition (T, δ) of G, there must exist a cut (X, Y) in (T, δ) so that $|X \cap A| \geq \left\lfloor \frac{|A|}{3} \right\rfloor$ and $|Y \cap B| \geq \left\lfloor \frac{|B|}{3} \right\rfloor$.*

Proof. For a single $S \subseteq V(G)$ it is well known that since T is a ternary tree with leaf set $V(G)$ there exists a cut (X_S, Y_S) in (T, δ) associated with an edge $(x_S, y_S) \in E(T)$ so that $|X_S \cap S| \geq \lfloor \frac{|S|}{3} \rfloor$ and $|Y_S \cap S| \geq \lfloor \frac{|S|}{3} \rfloor$. Consider the path in T starting in edge (x_A, y_A) and ending in edge (x_B, y_B). The cut associated with any edge on this path will satisfy the statement in the lemma. $\square$

Lemma 7. *Let G be a graph, P a non-trivial prime graph in a split decomposition of G and (T, δ) a branch decomposition of sm-width less than k. If P has vertex b such that $|\mathrm{act}(b : P)| \geq 3k$ and $|\mathrm{act}(N(b) : P)| \geq 9k$, then there must exist vertex $a \in N(b)$ such that a and b form a heavy pair.*

Proof. Assume for contradiction that this is not the case, and all vertices $v \in N(b)$ have $|\mathrm{act}(v : P)| < 3k$. By Lemma 6 applied to $A = \mathrm{act}(b : P)$ and $B = \mathrm{act}(N(b) : P)$, there must be a cut (X, Y) in (T, δ) so that $|\mathrm{act}(b : P) \cap X| \geq k$ and $|\mathrm{act}(N(b) : P) \cap Y| \geq 3k$. We first show that $\mathrm{act}(N(b) : P) \subseteq Y$ and thus $|\mathrm{act}(N(b) : P) \cap Y| \geq 9k$.

By Lemma 5, (X, Y) is a split. Since no $v \in N(b)$ has $|\mathrm{act}(v : P)| \geq 3k$ the fact that $|Y \cap \mathrm{act}(N(b) : P)| \geq 3k$ means that $\left|\mathrm{tot}^{-1}(Y : P)\right| \geq 2$. By Lemma 4, this means $X \subseteq \mathrm{tot}(b : P)$ and thus $\mathrm{tot}(N(b) : P) \subseteq V \setminus X = Y$ and thus $\mathrm{act}(N(b) : P) \subseteq Y$.

Again, by Lemma 6, applied to $\mathrm{act}(N(b) : P) = A = B$, there must exist a cut (X', Y') in (T, δ) so that $|X' \cap \mathrm{act}(N(b) : P)| \geq 3k$ and $|Y' \cap \mathrm{act}(N(b) : P)| \geq 3k$. Since we have already shown that $\mathrm{act}(N(b) : P) \subseteq Y$ and both (X, Y) and (X', Y') are cuts in (T, δ), either $X \subseteq X'$ or $X \subseteq Y'$. Without loss of generality, let us assume $X \subseteq X'$. Since there are at least $3k$ vertices of $\mathrm{act}(N(b) : P)$ in

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Y' and at least k vertices of $\text{act}(b : P)$ in $X \subset X'$, by Lemma 5 (X', Y') must be a split. The assumption that all $v \in N(b)$ have $|\text{act}(v : P)| < 3k$ means that $\text{tot}^{-1}(X' : P)$ and $\text{tot}^{-1}(Y' : P)$ contain at least two vertices from $\text{act}(N(b) : P)$, which together with the fact that (X', Y') is a split is a contradiction of Lemma 4. Therefore, our assumption was wrong, and there must exist vertex $a \in N(b)$ such that $|\text{act}(a : P)| \geq 3k$. $\qquad\square$

Lemma 8. *Let G be a graph, P a non-trivial prime graph in a split decomposition of G and (T, δ) a branch decomposition of sm-width less than k. For any heavy pair a, b in P we have that there must exist a cut (X, Y) in (T, δ) where $\text{tot}^{-1}(X : P) = \{a, b\}$ and $|N(X)| < k$.*

Proof. By the definition of a heavy pair, $|\text{act}(a : P)| \geq 3k$ and $|\text{act}(b : P)| \geq 3k$. We know, by Lemma 6 on $\text{act}(a : P)$ and $\text{act}(b : P)$ that there must exist a cut $(X_1, \overline{X_1})$ in (T, δ) where $|\text{act}(b : P) \cap X_1| \geq k$ and $|\text{act}(a : P) \cap \overline{X_1}| \geq k$. By Lemma 5, this cut must be a split, and by Lemma 4 either $X_1 \subseteq tot(b)$ or $\overline{X_1} \subseteq tot(a)$. Without loss of generality, let $X_1 \subseteq tot(b)$. This means that $\text{act}(a : P) \subseteq \text{tot}(a : P) \subseteq \overline{X_1}$. By Lemma 6 on $\text{act}(a : P)$, we know there is a cut $(X_2, \overline{X_2})$ in (T, δ) so that $|\text{act}(a : P) \cap X_2| \geq k$ and $|\text{act}(a : P) \cap \overline{X_2}| \geq k$. Since $\text{act}(a : P) \subseteq \overline{X_1}$ and both $(X_1, \overline{X_1})$ and $(X_2, \overline{X_2})$ are cuts in (T, δ), either $X_1 \subseteq X_2$ or $X_1 \subseteq \overline{X_2}$. Without loss of generality, let $X_1 \subseteq X_2$. This means $\overline{X_2}$ contains vertices of $\text{act}(a : P)$ and of $\text{act}(b : P)$, and thus $|\text{tot}^{-1}(X_2 : P)| \geq 2$. However, $(X_2, \overline{X_2})$ is a split and $\overline{X_2} \cap \text{tot}(a : P) \neq \emptyset$, so by Lemma 4, $\overline{X_2} \subseteq \text{tot}(a : P)$.

Now, let A and B be the largest sets $\overline{X_2} \subseteq A \subseteq \text{tot}(a : P)$ and $X_1 \subseteq B \subseteq \text{tot}(b : P)$, so that there exist cuts $(A, \overline{A})$ and $(B, \overline{B})$ in (T, δ) (an equivalent way of saying this is that $(B, \overline{B})$ is the cut associated with the last edge along the path in T from $(X_1, \overline{X_1})$ to $(X_2, \overline{X_2})$ so that $B \subseteq \text{tot}(b : P)$, and $(A, \overline{A})$ is the cut associated with the last edge from $(X_2, \overline{X_2})$ to $(X_1, \overline{X_1})$ where $A \subseteq \text{tot}(a : P)$). By the above, such sets must exist. Since T is a cubic tree, the edge in $E(T)$ representing $(B, \overline{B})$ must be adjacent to two other edges that represent two cuts $(R_1, B \cup R_2)$ and $(R_2, B \cup R_1)$ where $\overline{B} = R_1 \cup R_2$, as depicted in Figure 4. As $B \cap A$ is empty, and there is a cut $(A, \overline{A})$, we have either $A \subseteq R_1$ or $A \subseteq R_2$. Without loss of generality $A \subseteq R_2$. By maximality of $B \subseteq \text{tot}(b : P)$ we have $R_1 \setminus \text{tot}(b : P) \neq \emptyset$ and thus $|\text{tot}^{-1}(B \cup R_1 : P)| \geq 2$. However, $(B \cup R_1, R_2)$ must be a split by Lemma 5, since there are at least k vertices of $\text{act}(a : P)$ in $A \subseteq R_2$ and at least k vertices of $\text{act}(b : P)$ in $(B \cup R_1)$. Furthermore, by Lemma 4, $R_2 \subseteq \text{tot}(a : P)$. However, as $A \subseteq R_2$ is the largest set $\overline{X_2} \subseteq A \subseteq \text{tot}(a : P)$ so that $(A, \overline{A})$ is in (T, δ), we have $A = R_2$. This means the cut $(R_1, B \cup R_2)$ is in fact $(R_1, B \cup A)$ and thus $\text{tot}^{-1}(B \cup A : P_G) = \{a, b\}$.

Furthermore, since $|\text{tot}^{-1}(B \cup A : P)| = 2$ and $|V(P)| \geq 4$, by Lemma 4 $(R_1, B \cup A)$ cannot be a split. The MM-value of a cut is the same as the smallest vertex cover of the bipartite graph associated with cut. As all minimal vertex covers of $G[R_1, A \cup B]$ contain either at least k vertices of $\text{act}(b : P) \cap B$, at least k vertices of $\text{act}(a : P) \cap A$, or all the neighbors of A and B in R_1, we conclude that $N(A \cup B) < k$. $\qquad\square$
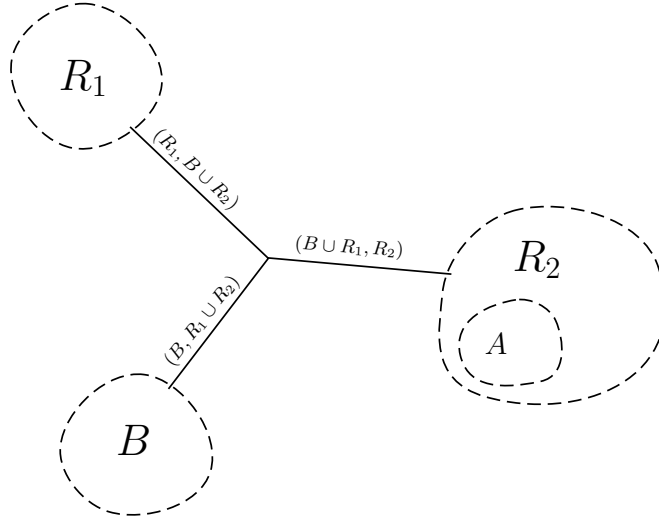
Figure 4: The two cuts incident to $(B, \overline{B})$ in the decomposition.

From Lemma 8 we see that each vertex is incident to at most one heavy edge. From this we deduce the following Corollary.

Corollary 9. *For a non-trivial prime graph, its heavy edges form a matching.*

Lemma 10. *Let $smw(G) < k$ and let P be a prime graph in a split decomposition of G and let $A \subseteq V(P)$ with $2 \le |A| \le |V(P)| - 2$. If no heavy edges cross the cut $(A, V(P) \setminus A)$ in P and $\mathrm{sm}(A) < t$ with respect to P, then $\mathrm{sm}(\mathrm{tot}(A : P))$ with respect to G is less than $9tk$.*

Proof. By König's theorem the size of a maximum matching and size of a minimum vertex cover in a bipartite graph is the same. Since P is a prime graph it has no non-trivial splits and thus for P we have $mm(V_i) = sm(V_i)$ for any $V_i \subseteq V(P)$. Thus if $\mathrm{mm}(A) < t$ in P, there must exist a vertex cover $C \subseteq V(P)$ for $P[A, V(P) \setminus A]$ of cardinality less than t. Based on the vertices of C we create a vertex cover $C' \subseteq V(G)$ for the subgraph $G[\mathrm{tot}(A : P), \overline{\mathrm{tot}(A : P)}]$ of G having cardinality less than $9tk$, proving that $\mathrm{mm}(\mathrm{tot}(A : P)) < 9tk$.

We start with $C' = \emptyset$ and add for each $v \in C$ a set C_v to C'. For any vertex $v \in C$ not part of any heavy pair in P, it follows from Lemma 7 and Lemma 8 that either $|\mathrm{act}(v : P)| < 3k$ or $|\mathrm{act}(N(v) : P)| < 9k$. In both cases, there is a set $C_v \subseteq V(G)$ of size at most $9k$ that we add to C' so that each edge of G incident with $\mathrm{tot}(v : P)$ (and in particular those crossing the cut $\left(\mathrm{tot}(A : P), \overline{\mathrm{tot}(A : P)} \right)$) has an endpoint in C'. If on the other hand v is part of a heavy pair uv in P, we note, by the assumption in the lemma, that u must be on the same side as v in the cut $(A, V(P) \setminus A)$ of P. Again, it follows from Lemma 8 that $|\mathrm{act}(N(\{u, v\}) : P)| < k$, so there is a set $C_v \subseteq V(G)$ of size at most k that we add to C' so that all edges in the subgraph $G[\mathrm{tot}(A : P), \overline{\mathrm{tot}(A : P)}]$ incident to

tot$(v : P)$ are covered by C'. Doing this for every vertex $v \in C$ will lead to a set C' with $|C'| \leq 9|C|k < 9tk$. Furthermore, by the definition of a vertex cover, and tot(), the set C' covers all the edges of $E(G[\text{tot}(A : P), \text{tot}(A : P)])$. $\square$

When deleting a vertex any cut that was a split remains a split or results in a cut with a single vertex on one side, and no new matchings are introduced.

Observation 11. *The sm-width of a graph G is at least as big as the sm-width of any induced subgraph of G.*

Theorem 12. *Let $smw(G) < k$ and let P be a prime graph in a split decomposition of G We can in $\mathcal{O}^*(8^k)$-time construct a branch decomposition (T'_P, δ'_P) of P so that for each cut (X, Y) of P induced by an edge of (T'_P, δ'_P), the cut $(\text{tot}(X : P), \text{tot}(Y : P))$ of G has sm-value less than $54k^2$.*

Proof. If P is a trivial prime graph, i.e. $|V(P) \leq 3|$, every cut (X, Y) of P is a split. This implies by the definition of a split decomposition that $(\text{tot}(X : P), \text{tot}(Y : P))$ in G also is a split of G. Hence, $\text{sm}(\text{tot}(X : P))$ of G equals one.

We now consider the case when P is non-trivial. Since P is isomorphic to an induced subgraph of G (this follows directly from definition of split decompositions) and $\text{smw}(G) < k$, by Observation 11, the sm-width of P is less than k. Also, since P by definition has no splits, we have $\text{mmw}(P) = \text{smw}(P) < k$. By Theorem 3 and Lemma 2, we can compute a branch decomposition (T_P, δ_P) of P with MM-width less than $3k$ in $\mathcal{O}^*(8^k)$-time. By a non-leaf edge of T_P we mean an edge with both endpoints an inner node of T_P. The cut in P induced by a non-leaf edge of (T_P, δ_P) will have at least two vertices on each side. We call such cuts non-leaf cuts of P induced by (T_P, δ_P). Note that cuts having one side containing a singleton $X = \{v\}$ are easy to deal with, either the singleton is a vertex of $V(G)$ and then $\text{tot}(X : P) = \{v\}$, or v is a marker and the cut $(\text{tot}(X : P), \text{tot}(Y : P))$ of G is a split, and thus in both cases $\text{sm}(\text{tot}(X : P)) = 1$. For the remainder we consider only non-leaf cuts.

Denote by $h(A)$ the number of heavy edges crossing the non-leaf cut $(A, V(P) \backslash A)$. If none of the non-leaf cuts of P induced by (T_P, δ_P) have heavy edges crossing them, i.e. $h(A) = 0$ for all non-leaf cuts, we apply Lemma 10 with $t = 3k$ and are done, getting for any cut (X, Y) of P induced by an edge of (T_P, δ_P) a bound of $\text{sm}(\text{tot}(X : P)) \leq 3k9k = 27k^2$. On the other hand, if some non-leaf cuts of P induced by (T_P, δ_P) do have heavy edges crossing them, we restructure the decomposition (T_P, δ_P) to a decomposition (T'_P, δ'_P) as follows: for each heavy pair a, b in $V(P)$ crossing such a non-leaf cut we remove the leaf in T_P mapping to b and make a new leaf mapping to b as sibling of the leaf mapping to a. By Corollary 9 the heavy edges in P form a matching, so this is easily done for all heavy edges of P crossing non-leaf cuts, without conflicts. Since all such heavy pairs are now mapped to leaves that are siblings of T'_P none of the non-leaf cuts of P induced by (T'_P, δ'_P) will have a heavy edge crossing them.

Let us look at how the restructuring altered the sm-value of non-leaf cuts. Note that for each non-leaf cut $(A', V(P) \setminus A')$ in (T'_P, δ'_P) there is an associated

non-leaf cut $(A, V(P) \setminus A)$ in (T_P, δ_P) with $h(A)$ heavy edges crossing this cut, such that we move between the two cuts by moving $h(A)$ vertices across the cut. We have $\mathrm{mm}(A') \leq \mathrm{mm}(A) + h(A)$, as the maximum matching of a cut can increase by at most one for each vertex moved over the cut. Moreover, by Corollary 9 the heavy edges in P form a matching, which means that $h(A) \leq \mathrm{mm}(A)$, implying $\mathrm{mm}(A') \leq 2\,\mathrm{mm}(A) \leq 2 \times 3k$. We can therefore apply Lemma 10 with $t = 6k$ and this means we have $\mathrm{sm}(\mathrm{tot}(A : P)) \leq 6k9k = 54k^2$. □

Theorem 13. *Given a graph G with $smw(G) < k$, we can compute a branch decomposition (T, δ) of G of sm-width less than $54k^2$ in $\mathcal{O}^*(8^k)$-time.*

Proof. For any G' in a split decomposition of G, be it a prime graph or a composition of prime graphs, we claim the following: We can create a branch decomposition $(T_{G'}, \delta_{G'})$ of G' so that for each cut $(A, \overline{A})$ in $(T_{G'}, \delta_{G'})$, the sm-value of the cut $(\mathrm{tot}(A : G'), \mathrm{tot}(\overline{A} : G'))$ in G is less than $54k^2$. We call this latter cut the cut induced in G. We will give a proof of this by induction on the number of splits in G':

In the case that G' does not have a split, it must be a prime graph and by Theorem 12 the hypothesis holds. If on the other hand G' does have a split, it must be decomposed by two subgraphs G_1 and G_2 in the split decomposition of G sharing a single marker v. By induction, these two graphs have such branch decompositions (T_{G_1}, δ_{G_1}) and (T_{G_2}, δ_{G_2}) inducing cuts in G where the sm-value is less than $54k^2$. We will now merge these two decompositions into a decomposition $(T_{G'}, \delta_{G'})$ for G'. What we do is to identify the vertex v in T_{G_1} and v in T_{G_2} ($V(T_{G'}) = V(T_{G_1}) \cup V(T_{G_2})$ and $E(T_{G'}) = E(T_{G_1}) \cup E(T_{G_2})$). By also joining the mapping functions δ_{G_1} and δ_{G_2} in the natural way, we get a branch decomposition $(T_{G'}, \delta_{G'})$ of G' where each cut induced in G is already a cut induced in G by either (T_{G_1}, δ_{G_1}) or (T_{G_2}, δ_{G_2}). By induction (T_{G_1}, δ_{G_1}) and (T_{G_2}, δ_{G_2}) only have cuts where the sm-value of all induced cuts is less than $54k^2$, so the same holds for $(T_{G'}, \delta_{G'})$.

The recursive algorithm resulting from the above induction has a runtime of $\mathcal{O}^*(8^k)$ on each prime graph and no more than linear time of work (finding the marker) in all other partially decomposed graphs in the decomposition, totalling to a runtime of $\mathcal{O}^*(8^k)$ since the number of prime graphs in a split decomposition is polynomial in n. □

4 Dynamic programming parameterized by sm-width

In this section we solve MaxCut, Graph Coloring, Hamiltonian Cycle and Edge Dominating Set on a graph G by a bottom-up traversal of a rooted branch decomposition (T, δ) of G, in time FPT parameterized by the sm-width of (T, δ). Previously, we did not define the tree T to be rooted, but this will help guide the algorithm by introducing parent-child relationships. To root T,

we first pick any edge of T and subdivide it. We then root the tree in the newly introduced vertex, resulting in a rooted binary tree consisting of the exact same cuts as the original decomposition.

In the bottom-up traversal of the rooted tree we encounter two disjoint subsets of vertices $A, B \subseteq V(G)$, as leaves of two already processed subtrees, and need to process the subtree on leaves $A \cup B$. There are three cuts of G involved: $(A, \overline{A})$, $(B, \overline{B})$, $(A \cup B, \overline{A \cup B})$, and each of them can be of type split, or of type non-split (also called type mm for maximum-matching). This gives six cases that need to be considered, at least if we use the standard framework of table-based dynamic programming. We instead use an algorithmic framework for decision problems where we JOIN sets of certificates while ensuring that the result preserves witnesses for a 'yes' instance. Under this framework, the algorithm for MAXCUT becomes particularly simple, and only two cases need to be handled in the JOIN, depending on whether the 'parent cut' $(A \cup B, \overline{A \cup B})$ is a split or not. For the other three problems we must distinguish between the two types of 'children cuts' in order to achieve FPT runtime, and the algorithms are more complicated.

Let us describe the algorithmic framework. As usual, e.g. for problems in NP, a verifier is an algorithm that given a problem instance G and a certificate c, will verify if the instance is a 'yes'-instance, and if so we call c a witness. For our algorithms we will use a commutative and associative function $\oplus(x, y)$, that takes two certificates x, y and creates a set of certificates. This is extended to sets of certificates X_A, X_B by $\oplus(X_A, X_B)$ which creates the set of certificates $\bigcup_{x_A \in X_A, x_B \in X_B} \oplus(x_A, x_B)$. For a graph decision problem, an input graph G, and any $X \subseteq V(G)$ we define $\mathrm{cert}(X)$ to be a set of certificates on only a restricted part of G, which must be subject to the following constraints:

- If G is a 'yes'-instance, then $\mathrm{cert}(V(G))$ contains a witness.

- For disjoint $X, Y \subseteq V(G)$ we have $\oplus(\mathrm{cert}(X), \mathrm{cert}(Y)) = \mathrm{cert}(X \cup Y)$.

For FPT runtime we need to restrict the size of a set of certificates, and the following will be useful. For $X \subseteq V(G)$ and certificates $x, y \in \mathrm{cert}(X)$, we say that x *preserves* y if for all $z \in \mathrm{cert}(\overline{X})$ so that $\oplus(y, z)$ contains a witness, the set $\oplus(x, z)$ also contains a witness. We denote this as $x \preceq_X y$. A set S preserves $S' \subseteq \mathrm{cert}(X)$, denoted $S \preceq_X S'$, if for every $x' \in S'$ there exists a $x \in S$ so that $x \preceq_X x'$. A certificate $x \in \mathrm{cert}(X)$ so that there exists a $y \in \mathrm{cert}(\overline{X})$ where $\oplus(x, y)$ contains a witness, is called an *important* certificate.

For a rooted branch decomposition (T, δ) of a graph G and vertex $v \in V(T)$, we denote by V_v the set of vertices of $V(G)$ mapped by δ from the leaves of the subtree in T rooted at v. With these definitions we give a generic recursive (or bottom-up) algorithm called RECURSIVE that takes (T, δ) and a vertex w of T as input and returns a set $S \preceq_{V_w} \mathrm{cert}(V_w)$, as follows:

- at a leaf w of T INITIALIZE and return the set $\mathrm{cert}(\{\delta(w)\})$

- at an inner node w first call RECURSIVE on each of the children nodes a and b and then run procedure JOIN on the returned input sets S_1, S_2 of

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certificates, with $S_1 \preceq_{V_A}$ cert(V_a) and $S_2 \preceq_{V_b}$ cert(V_b), and return a set $S \preceq_{V_a \cup V_b} \oplus(S_1, S_2)$

- at the root we will have a set of certificates $S \preceq_{V(G)}$ cert$(V(G))$

Calling RECURSIVE on the root r of T and running a verifier on the output solves any graph decision problem in NP. Correctness of this procedure follows from the definitions. The extra time spent by the verifier is going to be $\mathcal{O}^*(|S|)$, and for an FPT algorithm we will require that all $|S|$ be $\mathcal{O}^*(f(k))$, i.e. FPT in the sm-width k of (T, δ).

In the following subsections we show how to solve each of the respective four problems in FPT time. A rough sketch of the idea of how this can be achieved for each of the problems is shown below. A formal definition of each of the problems is given in each of their respective subsections.

Maximum Cut. MAXCUT is the one out of the four problems which has the most simple algorithm. To compute a maximum cut, we will give an algorithm to solve t-MAXCUT, which instead of maximizing a cut asks for a cut of size at least t. Running t-MAXCUT for increasing values of t, will determine the size of a maximum cut. The certificates for this problem is subsets of vertices and a witness is a subset S so that the number of edges with one endpoint in S and one in $V(G) \setminus S$ is at least t (i.e., witnesses are cuts of size at least t). We show that for a cut $(A, \overline{A})$ and subsets S_1 and S_2 in cert(A), if the neighbourhood of S_1 and S_2 in $\overline{A}$ are the same, then one of the sets preserves the other in A. As this number is bounded by 2 and $2^{\mathrm{mm}(A)}$, for split and non-split cuts, respectively, we will be able to give an FPT algorithm for solving MAXCUT.

Hamiltonian Cycle. For HAMILTONIAN CYCLE, certificates are disjoint paths or cycles, and a witness is a Hamiltonian cycle. The important information is what neighbourhood the endpoints of each path has over the cut. For each certificate we keep track of the number of *path classes*, which are sets of paths with the same neighborhood over the cut, and the size of each such path class. The total number of path classes over all certificates is also important. For a split cut, the size of a class might be anything from 1 to n, but there will be only one class in total. For a non-split cut of sm-value k, the total number of path classes is bounded by 2^{2k} and since each path is vertex disjoint the number of paths in any important certificate is bounded by k. Based on this the JOIN operation will be able to find a FPT-sized set of certificates preserving a full set.

Chromatic Number. For CHROMATIC NUMBER, we will actually solve t-COLORING, which asks whether the input graph can be colored by at most t colors, and from this conclude that CHROMATIC NUMBER can be solved in the same time when excluding polynomial factors. We note that a graph of sm-width k, unlike graphs of treewidth k, may need more than $k + 1$ colors. We let all partitions into t parts where the parts induce independent sets be our certificates. What matters for a certificate is what kind of certificates it can be combined with

to yield a new certificate, i.e. inducing an independent set also across the cut. For non-split cuts, this means the number of important certificates is bounded by the number of ways to t-partition the vertices in the k-vertex cover of the cut, which is a function of k. For a split cut, what is important is the number of parts of a partition/certificate that have neighbors across the cut. The certificate minimizing this number will preserve all other certificates. Based on this the JOIN operation will be able to find a preserving set of certificates of FPT-size.

Edge Dominating Set. For EDGE DOMINATING SET (or t-EDGE DOMINAT-ING SET which is what we actually solve) the certificates are subgraphs of G and a witness is a graph $G' = (V', E')$ so that each vertex in V' is incident with an edge in E', and E' is an edge dominating set of G of size at most t. The idea of how to make an FPT JOIN-procedure is that for a vertex cover C of a cut, the number of ways a certificate can project to C is limited by a function of the size of C. Based on this we find a preserving set of FPT cardinality when $|C|$ is at most k. When $|C|$ is not bounded by k, we have a split. For splits we limit the max number of certificates needed for a preserving set by a polynomial of n. This is because almost all edges on one side of the cut affect the rest of the edges uniformly, and the other way around.

4.1 Maximum Cut

4.1.1 The Problem.

The problem t-MAXCUT asks, for a graph G, whether there exists a set $W \subseteq V(G)$ so that the number of edges in $G[W, \overline{W}]$ is at least t. For a set X, we denote by $\delta_G(X)$ the number of edges in $G[V(G) \cap X, V(G) \setminus X]$ (note that X does not need to be a subset of $V(G)$).

4.1.2 The certificates and $\oplus$.

For t-MAXCUT, we define cert(X) for $X \subseteq V(G)$ to be all the subsets of X, and we define $\oplus(x, y)$ to be the union function; $\oplus(x, y) = \{x \cup y\}$. We solve t-MAXCUT by use of RECURSIVE and the below procedure JOIN$_{maxcut}$ with input specification as described above.

4.1.3 The Join$_{maxcut}$ function.

Procedure JOIN$_{maxcut}$
   Input: $S_1 \preceq_{V_a} cert(V_a)$ and $S_2 \preceq_{V_b} cert(V_b)$ with $A = V_a \cup V_b$
   Output: $S \preceq_A \oplus(\text{cert}(V_a), \text{cert}(V_b)) = \text{cert}(A)$

$S' \leftarrow \{s_1 \cup s_2 : s_1 \in S_1, s_2 \in S_2\}$ /* note $S' = \oplus(S_1, S_2)$ */
$S \leftarrow \emptyset$
$C \leftarrow$ a minimum vertex cover of $G[A, \overline{A}]$
if $(A, \overline{A})$ is a split **then for** $z = 0, \ldots, n$ **do**
    $c' \leftarrow \text{argmax}_{c \in S'}\{\delta_{G[A]}(c) : |N(\overline{A}) \cap c| = z\}$

$\qquad S \leftarrow S \cup \{c'\}$
else for all subsets $S_C \subseteq C$ **do**
$\qquad c' \leftarrow \operatorname{argmax}_{c \in S'} \left\{ \delta_{G[A]}(c) : S_C \cap A = c \cap A \right\}$
$\qquad S \leftarrow S \cup \{c'\}$
return S

Lemma 14. *Procedure* JOIN_{maxcut} *is correct and runs in time* $\mathcal{O}^*((|S_1||S_2|)2^k)$, *producing a set* S *of cardinality* $\mathcal{O}(n + 2^k)$.

Proof. We see that $S' \preceq_A \operatorname{cert}(A)$, since $S' = \oplus(S_1, S_2) \preceq_A \operatorname{cert}(A)$, and S' can be calculated in time $\mathcal{O}^*(|S_1||S_2|)$. Finding a vertex cover of a $G[A, \overline{A}]$ can be done in polynomial time, since $G[A, \overline{A}]$ is a bipartite graph. Also, when $(A, \overline{A})$ is not a split, then $\operatorname{mm}(A) \leq k$ and $|C| \leq k$. Combined with a polynomial amount of work for each iteration of the for loops, and loops iterating at most $\max\{n, 2^k\}$ times (making the size of S also bounded by $n + 2^k$), the total runtime is $\mathcal{O}^*(|S| \, 2^k)$.

To show that $S \preceq_A S'$ (and thus also $S \preceq_A \operatorname{cert}(A)$) we have to make sure that if there exists a witness x of t-MaxCut so that for $x_A \in S'$ and $x_{\overline{A}} \in \operatorname{cert}(\overline{A})$ we have $\{x\} \subseteq \oplus(x_A, x_{\overline{A}})$, then there must exist a certificate $x' \in S$ so that $x' \preceq_A x_A$. We assume there exists such a witness x with x_A and $x_{\overline{A}}$ defined as above. We have two cases to consider; when $(A, \overline{A})$ is a split, and when it is not.

We first consider the case when $(A, \overline{A})$ is a split. Since x is a witness, $\delta_G(x) \geq t$. Let $z = |N(\overline{A}) \cap x|$. We have $\delta_G(x) = \delta_{G[A]}(x_A) + \delta_{G[\overline{A}]}(x_{\overline{A}}) + \delta_{G[A,\overline{A}]}(x)$, and $\delta_{G[A,\overline{A}]}(x) = |N(\overline{A}) \cap x_A| \times |N(A) \setminus x_{\overline{A}}| = z |N(A) \setminus x_{\overline{A}}|$. Since S contains $c \in S'$ maximizing $\max_{c \in S'} \{\delta_{G[A]}(c) : |N(\overline{A} \cap c)| = z\}$, we have $\delta_G(\oplus(c, x_{\overline{A}})) = \delta_G(x) + \delta_{G[A]}(c) - \delta_{G[A]}(x_A) \geq \delta_G(x)$ meaning $\oplus(c, x_{\overline{A}})$ is a witness, and so $S \preceq_A S'$.

Now, consider the case when $(A, \overline{A})$ is not a split (this means $\operatorname{mm}(A) \leq k$). Let C be the vertex cover used in the procedure and x_C be $x \cap C$. As C disconnects A and $\overline{A}$, we have $\delta_G(x) = \delta_{G[A]}(x_A) + \delta_{G[\overline{A}]}(x_{\overline{A}}) + \delta_{G[A,\overline{A}]}(x_C)$. We notice that for all $c \in S'$ so that $c \cap C = x_A \cap C$, we have $\delta_{G[A,\overline{A}]}(\oplus(c, x_{\overline{A}})) = \delta_{G[A,\overline{A}]}(x_C)$. Therefore, as S contains the certificate c of S' where c maximizes $\max_{c \in S'} \{\delta_{G[A]}(c) : x_A \cap C = c \cap A\}$, we must have that $\delta_G(\oplus(c, x_{\overline{A}})) \geq \delta_G(x)$. So $\oplus(c, x_{\overline{A}})$ is also a witness, and hence $S \preceq S'$. $\qquad \square$

Theorem 15. *Given a graph* G *and branch decomposition* (T, δ) *of sm-width* k, *we can solve* MaxCut *in time* $\mathcal{O}^*(8^k)$.

Proof. In Lemma 14 we show JOIN_{maxcut} is correct and produce a preserving set S of size at most $\mathcal{O}^*(2^k)$ in time $\mathcal{O}^*(|S_1||S_2|2^k)$. So, using RECURSIVE with JOIN_{maxcut}, we know the size of both of the inputs of JOIN_{maxcut} is at most the size of its output, i.e., $|S_1|, |S_2| \leq \mathcal{O}^*(2^k)$. So, each call to RECURSIVE has runtime at most $\mathcal{O}^*(8^k)$. As there are linearly many calls to RECURSIVE and there is a polynomial time verifier for the certificates RECURSIVE produces, by the definition of $\preceq$, the total runtime for solving t-MaxCut is also bounded by

$\mathcal{O}^*(8^k)$. To solve MaxCut, we run the t-MaxCut algorithm for all values of $t \leq n^2$, and hence have the same runtime when excluding polynomials of n. $\quad\square$

4.2 Hamiltonian Cycle

4.2.1 The problem.

For a graph G, a subgraph G' of G where G' is a cycle, we say that G' is a *hamiltonian cycle* of G if $V(G') = V(G)$. The decision problem Hamiltonian Cycle asks, for an input graph G, whether there exists a hamiltonian cycle of G.

4.2.2 The certificates and $\oplus$.

We notice for a set $A \subseteq V(G)$ and hamiltonian cycle G' of G that $G'[A]$ is either the hamiltonian cycle itself (if $A = V(G)$) or a set of vertex disjoint paths and isolated vertices. For ease of notation, we will throughout this section regard isolated vertices as paths (of length zero). That is, a certificate $G'[A]$ is always either a set of vertex disjoint paths or a cycle. Based on this observation, it is natural to let $\text{cert}(X)$ for $X \subseteq V(G)$ on the problem Hamiltonian Cycle be all subgraphs G' of G so that $V(G') = X$ and G' consists only of disjoint paths or of a cycle of length $|V(G)|$. The witnesses of $\text{cert}(V(G))$ are exactly the certificates that are hamiltonian cycles of G. Clearly a polynomial time verifier exists, as we can easily confirm, in polynomial time, that a hamiltonian cycle in fact is a hamiltonian cycle. Also, as $\text{cert}(V(G))$ contains all hamiltonian cycles of $V(G)$, it must contain a witness if G is a 'yes'-instance.

For disjoint sets $A, B \subset V(G)$, $G_x \in \text{cert}(A)$, and $G_y \in \text{cert}(B)$, we define $\oplus(G_x, G_y)$ to be the set of all certificates $G_z = (A \cup B, E(G_x) \cup E(G_y) \cup E')$ where E' is a subset of the edges crossing (A, B). That is, $\oplus(G_x, G_y)$ is the set of all graphs generated by the disjoint union of G_x and G_y and adding edges from G with one endpoint in A and one endpoint in B that are also valid certificates. This is a valid definition for $\oplus$, as we have $\text{cert}(A \cup B) = \oplus(\text{cert}(A), \text{cert}(B))$.

4.2.3 The Join$_{HC}$ function.

In Join$_{maxcut}(S_1, S_2)$ we first calculated $S = \oplus(S_1, S_2)$, and later reduced the size of S. However, by our definition of $\oplus$ for Hamiltonian Cycle, even for certificate sets S_1, S_2 of restricted cardinality, the set $\oplus(S_1, S_2)$ might be huge. Therefore, in Join$_{HC}$ we cannot allow to always run $\oplus$ inside our algorithm. Instead we will for each pair of certificates in S_1 and S_2 construct a set $S' \preceq \oplus(S_1, S_2)$ where $|S'|$ is bounded by an FPT function of n and k while possibly $S' \subset \oplus(S_1, S_2)$.

Before we present the algorithm, we need to introduce a few key observations and claims and give some new terminology.

For a certificate $G' \in \text{cert}(A)$ for $A \subset V(G)$, each path P of G' can be categorized by an unordered pair (N_1, N_2) so that for its two endpoints v_1 and v_2 (or single endpoint $v_1 = v_2$ if P is an isolated vertex) we have $N_1 = N(v_1) \setminus A$ and

$N_2 = N(v_2) \setminus A$. We say that two paths are from the same *class* of paths if they get categorized by the same unordered pair. Two certificates $G', G'' \in \text{cert}(A)$ are *path equivalent* if there exists a bijection σ between the paths of G' and the paths in G'' so that for each pair of paths $P \in G'$ and $\sigma(P) \in G''$, the path P is in the same path class as $\sigma(P)$.

Claim 16. *For disjoint sets $A, B \subset V(G)$ and certificates $G_A \in \text{cert}(A)$, $G_B \in \text{cert}(B)$, where G_A consists of z_A paths and G_B consists of z_B paths, we can compute $\oplus(G_A, G_B)$ in time $\mathcal{O}^*(2^{4z_A z_B})$.*

Proof. From the definition $\oplus(G_A, G_B)$ contains all valid certificates G' where $G' = G_A \cup G_B + E'$ where E' consists of edges crossing (A, B). As G' must be a valid certificate, each vertex must have degree at most 2, so each vertex of $A \cup B$ incident with an edge in E' must have degree at most 1. As G_A and G_B consist of only paths, the vertices of degree at most 1 are exactly the vertices that occur as an endpoint of a path in either G_A or G_B. Therefore, E' must be a subset of the edges going from the at most $2z_A$ endpoints in G_A to the at most $2z_B$ endpoints in G_B. The number of such subsets is bounded by $2^{(2z_A 2z_B)}$, and finding such a set we can do with a runtime of no more than a polynomial factor larger than the size of this set. $\square$

Claim 17. *For any subset $A \subset V(G)$ and certificates $G_1, G_2 \in \text{cert}(A)$, we have $G_1 \preceq_A G_2$ if G_1 is path equivalent to G_2.*

Proof. Suppose there is a certificate $G_3 \in \text{cert}(\overline{A})$ and witness $W \in \oplus(G_3, G_2)$. That means that for a set of edges $E_W \subseteq E(G[A, \overline{A}])$ we have $W = G_2 \cup G_3 + E_w$. From the definition of path classes and path equivalence, there exists a bijection σ from paths of G_2 to paths in G_1 so that for each path P in G_2 and edges $a_1 p_1, a_2 p_2 \in E_W$ so that p_1, p_2 are the endpoints of P, there must exist two edges $a_1 p_1', a_1 p_2' \in E(G[A, \overline{A}])$ where p_1', p_2' are the endpoints of $\sigma(P)$ in G_1. Thus, if replacing the edges E_W with these edges, and replacing each path P in G_2 with the path $\sigma(P)$ of G_3, we have an hamiltonian cycle. So, if $\oplus(G_2, G_3)$ contains a witness, so must $\oplus(G_1, G_3)$. $\square$

Lemma 18. *For $A \subset V(G)$, if G' is an important certificate of $\text{cert}(A)$, then the number of paths in G' is at most $\text{mm}(A)$.*

Proof. Since G' is an important certificate, there must exist a certificate $G'' \in \text{cert}(\overline{A})$ so that $\oplus(G', G'')$ contains a witness. This means the paths of G' and G'' can be joined together by edges E' from $G[A, \overline{A}]$ to form a hamiltonian cycle C. If we direct the cycle C, each path of G' and of G'' must be incident with exactly one in-edge and one out-edge. By looking at the edges in E' going from A to $\overline{A}$, we see that these edges make a matching of $G[A, \overline{A}]$, concluding that the number of paths in G' is at most the size of a maximum matching, i.e., $mm(A)$. $\square$

For a certificate $G' \in \text{cert}(A)$ and path P of G', we say that P is an *isolated* path if one of its endpoints is not incident with an edge in $E(G[A, \overline{A}])$. That is, P is an isolated path if it is categorized by a pair containing an empty set.

As each vertex of an hamiltonian cycle has degree exactly two, and for two certificates $G_1 \in \text{cert}(A)$, $G_2 \in \text{cert}(\overline{A})$, each of the edges in certificate $G' \in \oplus(G_1, G_2)$ is either in $E(G_1)$, $E(G_2)$, or $E(G[A, \overline{A}])$, we get the following observation.

Observation 19. *If G' is an important certificate, G' can not contain any isolated paths.*

When computing $\oplus(p, q)$ for certificates p and q of A_1 and A_2 that are both splits, we know that the number of classes of paths in p and q is constant, since all paths are categorized by the same pair. This enables us to bound the number of ways needed to combine p and q in order to represent $\oplus(p, q)$, since we know a lot of them will be redundant. When, on the other hand, one of the two sets, for instance A_1, has $\text{mm}(A_1) \leq k$, then all paths going from A_1 to $\overline{A_1}$ must go through a separator of size $\leq k$. Together with Observation 19 this implies that each important certificate in $\text{cert}(A_1)$ can contain at most k paths. Using this we will again be able to reduce the number of possible combinations of p and q necessary to compute in order to get a set $S \preceq_A \oplus(p, q)$.

Procedure JOIN_{HC}(on node w with children a, b and $A_1 = V_a, A_2 = V_b$
                and $A = A_1 \cup A_2$ and given $S_1 \preceq_{A_1} cert(A_1)$ and
                $S_2 \preceq_{A_2} cert(A_2)$)

// *Generating $S \preceq_A \text{cert}(A)$*
$S \leftarrow \emptyset$
for each pair (G_1, G_2) in $S_1 \times S_2$ **do**
    add $G_1 \cup G_2$ to S
    $P_1, P_2 \leftarrow$ the sets of paths in G_1 and G_2, respectively
    $V_1, V_2 \leftarrow$ the sets of endpoints of P_1 and P_2, respectively
    if $|P_1| > \text{mm}(A_1)$ **or** $|P_2| > \text{mm}(A_2)$ **then continue**

    if neither $(A_1, \overline{A_1})$ nor $(A_2, \overline{A_2})$ is a split **then**
        add to S the set $\{G_1 \cup G_2 + E' : E' \subseteq E(G[V_1, V_2])\}$

    else if both of $(A_1, \overline{A_1})$ and $(A_2, \overline{A_2})$ are splits **then**
        for integers $1 \leq z \leq z' \leq \min\{|P_1|, |P_2|\}$ **do**
            $P' \leftarrow$ result of connecting z' paths in P_1 and z' paths in P_2
                together by edges crossing (A_1, A_2) to form z new paths
            add to S the subgraph $P_1 \cup P_2 \cup P'$

    else //*exactly one of $(A_1, \overline{A_1})$ and $(A_2, \overline{A_2})$ is a split*
        $s \leftarrow$ either 1 or 2, so that $(A_s, \overline{A_s})$ is a split
        $r \leftarrow 3 - s$ // *the index opposite of s*
        remove all but $2|P_r|$ of G_s's paths from P_s
        $V_s' \leftarrow$ the set of endpoints of the now smaller set P_1 of paths
        add to S all of $\{(G_1 \cup G_2) + E' : E' \subseteq E(G[V_s, V_r])\}$

// *Reducing the size of S*

> remove from S all certificates that are invalid, contain isolated paths, or
> contain more than $\text{mm}(A)$ paths

for $G_1, G_2 \in S$ **do**
   remove G_2 from S if G_1 and G_2 are path equivalent
return S

Lemma 20. *Procedure* JOIN_{HC} *is correct and runs in time* $\mathcal{O}^*(|S_1|^2 |S_2|^2 2^{16k^2})$, *producing a set* S *of cardinality* $\mathcal{O}(n + 4^{k^2})$.

Proof. We will first give a proof of the correctness of the algorithm, and then give an analysis of the runtime. To prove the correctness, we must show that for every important certificate $G_x \in \oplus(S_1, S_2)$, there exists a certificate $G'_x \in S$ so that $G'_x \preceq_A G_x$, and give a bound on the size of S.

As the algorithm iterates through all pairs $G_a, G_b \in S_1 \times S_2$, we know that for some G_a, G_b, we have $G_x \in \oplus(G_a, G_b)$. Now, let us look at the iteration of the algorithm where $G_1 = G_a$ and $G_2 = G_b$. That is, $G_x \in \oplus(G_1, G_2)$. The cuts $(A_1, \overline{A_1})$ and $(A_2, \overline{A_2})$ can either both be splits, be one split and one non-split, or both be non-splits.

The algorithm will set V_1 and V_2 to be the set of vertices in G_1 and G_2, respectively, of degree at most one. As each vertex in a (hamiltonian) cycle has degree exactly two, no vertex can have degree three or more in an important certificate. This means that for the case when neither $(A_1, \overline{A_1})$ nor $(A_2, \overline{A_2})$ is a split, when the algorithm adds $\{G_1 + G_2 + E' : E' \subseteq E(G[V_1, V_2])\}$, it is clear that this set preserves $\oplus(G_1, G_2)$.

Now suppose exactly one of $(A_1, \overline{A_1})$ and $(A_2, \overline{A_2})$ is a split. Without loss of generality, let $(A_2, \overline{A_2})$ be the split. That is, in the algorithm we have $A_1 = A_s$ and $A_2 = A_r$ in the algorithm. Notice that as each path has two endpoints, for G_x to be a valid and important certificate, the total number of paths in G_1 where at least one of its endpoints is incident with an edge in $E(G[V_1, V_2])$ in G_x is at most $2|P_2|$. Furthermore, as each path of G_1 is in the same path class, since $(A_1, \overline{A_1})$ is a split, it does not matter exactly which particular $\leq 2|P_2|$ paths of G_1 gets incident with an edge of $E(G[V_1, V_2])$. So, when the algorithm removes paths from P_1, since at least $2|P_2|$ of them remain (or all of them, if $|P_1| \leq 2|P_2|$), the set $\{G_1 + G_2 + E' : E' \subseteq E(G[V'_1, V_2])\}$ where V'_2 is the set of endpoints of the shrinked set of paths P_1, preserves $\oplus(G_1, G_2)$.

For the final case, when both $(A_1, \overline{A_1})$ and $(A_2, \overline{A_2})$ are splits, we see that each path of G_x can be one of at most three types of paths: (1) $(N(A_1) \setminus A, N(A_1) \setminus A)$, (2) $(N(A_2) \setminus A, N(A_2) \setminus A)$, or (3) $(N(A_1) \setminus A, N(A_2) \setminus A)$. From this we can conclude that a valid certificate G'_x of $\text{cert}(A)$ preserves G_x if G'_x contains exactly the same number of paths as G_x from each of the three path types mentioned. Let π_1, π_2 and π_3 be the number of paths in G_x of type (1),(2), and (3), respectively. The algorithm iterates through all integer values of z and $z' \geq z$ between 1 and $\min\{|P_1|, |P_2|\}$. In particular, at one iteration, $z = \pi_3$ and $z' = |P_1| - \pi_1 = |P_2| - \pi_2$. Therefore, connecting z' of the paths in $|P_1|$ and $|P_2|$ together to form $z = \pi_3$ new paths of type (3), we have in effect generated a

certificate preserving G_x. For the case when $\pi = 0$, the algorithm will not iterate through $z = 0$. However, in this case $G_x = G_1 \cup G_2$ which we have already added to S in the very start of the main loop for G_1 and G_2.

From this, we can conclude that the set S generated before the last part of the algorithm, where we reduce its size, preserves $\oplus(S_1, S_2)$. As we have already shown that path equivalent certificates preserve each other, the last step of reducing the size of S is going to maintain the fact that $S \preceq_A \oplus(S_1, S_2)$.

At the end of the algorithm, there are no invalid certificates in S, or certificates containing isolated paths. Also, no two certificates in S are path equivalent, so we have the following regarding the size of S: When $(A, \overline{A})$ is a split, there are at most n path classes, so the size of S is bounded by n. When $(A, \overline{A})$ is not a split, we know each certificate contains at most $\mathrm{mm}(A)$ paths. Furthermore, as the minimum vertex cover of a bipartite graph is of the same size as the maximum matching of the same graph, there is a vertex cover C of $G[A, \overline{A}]$ of size at most $\mathrm{mm}(A)$. This means that each neighbourhood over A ($N(S) \setminus A$ for some set $S \subseteq A$) is one of at most $2^{\mathrm{mm}(A)}$ possibilities. This means that each path can be represented by one of at most $(2^{\mathrm{mm}(A)})^2$ pairs of neighbourhoods. As each certificate contain at most $\mathrm{mm}(A)$ paths, we can conclude that the cardinality of S is no more than $((2^{\mathrm{mm}(A)})^2)^{\mathrm{mm}(A)} \leq 4^{k^2}$.

For the runtime of the algorithm, we notice that the first main loop runs at most $|S_1| \times |S_2|$ times. For each iteration we do at most $\mathcal{O}(2^{8k^2})$ operations (worst case is when exactly one of $(A_1, \overline{A_1})$ and $(A_2, \overline{A_2})$ is a split – then $|V_s'| \leq 4k$ and $|V_r| \leq 2k$). So, we do at most $\mathcal{O}^*((|S_1| \cdot |S_2|)2^{8k^2})$ operations on the first part. For the latter part, the size of S is bounded by the runtime of creating it, i.e., S is bounded by $\mathcal{O}^*((|S_1| \cdot |S_2|)2^{8k^2})$. So, as the algorithm in the end iterates through all pairs of elements in S, the final runtime is $\mathcal{O}^*((|S_1|^2 |S_2|^2)2^{16k^2})$. $\square$

Theorem 21. *Given a graph G and branch decomposition (T, δ) of sm-width k, we can solve* HAMILTONIAN CYCLE *in time* $\mathcal{O}^*(2^{24k^2})$.

Proof. In Lemma 20 we show the procedure JOIN_{HC} is correct and runs in time $\mathcal{O}^*(|S_1|^2 |S_2|^2 2^{16k^2})$, producing a set S of cardinality $\mathcal{O}(n + 4^{k^2})$. So, using RECURSIVE with JOIN_{HC}, we know the size of both of the inputs of JOIN_{HC} is at most the size of its output, i.e., $|S_1|, |S_2| \leq \mathcal{O}^*(4^{k^2})$. So, each call to RECURSIVE has runtime at most $\mathcal{O}^*(2^{24k^2})$. As there are linearly many calls to RECURSIVE and there is a polynomial time verifier for the certificates RECURSIVE produces, by the definition of $\preceq$, the total runtime is also bounded by $\mathcal{O}^*(2^{24k^2})$. $\square$

4.3 t-Coloring

4.3.1   The problem.

The decision problem t-COLORING asks for an input graph G, whether there exists a labelling function c of the vertices of $V(G)$ using only t colors in such a way that no edge has its endpoints labelled with the same color. Equivalently, it

asks whether there exists a t-partitioning of the vertices so that each part induces an independent set. For simplicity, we will allow empty parts in a partition (e.g., $\{\{x_1, x_2, x_3\}, \{x_4\}, \emptyset, \emptyset, \emptyset\}$ is a 5-partition of $\{x_1, x_2, x_3, x_4\}$).

For a set A and partition $p = \{p_1, p_2, \ldots, p_j\}$, we denote by $A \cap p$ the partition $\{p_1 \cap A, p_2 \cap A, \ldots, p_j \cap A\}$.

4.3.2 The certificates and $\oplus$.

For t-COLORING, we define $\text{cert}(X)$ for $X \subseteq V(G)$ to be all t-partitions of X and $\oplus(p, q)$ for $p = p_1, \ldots, p_t$ and $q = q_1, \ldots, q_t$ to be the following set of partitions

$$
\left\{ \bigcup_{i \leq t} \{\{p_i \cup q_{\sigma(i)}\}\} : \sigma \text{ is a permutation of } \{1, \ldots, t\} \right\} .
$$

This satisfies the constraint $\oplus(\text{cert}(X), \text{cert}(Y)) = \text{cert}(X \cup Y)$, so it is a valid definition of $\oplus$.

We can easily construct a polynomial time algorithm that given a t-partition p of independent sets (which there must exist at least one of if G is a 'yes'-instance) is able to confirm that G is a 'yes'-instance. So t-partitions of $V(G)$ forming independent sets will be our witnesses.

4.3.3 The Join_{col} function.

The main observation for the design of this procedure is that whenever $(A, \overline{A})$ is a split, there exists a single element $x \in \text{cert}(A)$ so that $\{x\} \preceq_A \text{cert}(A)$. Also, when $\text{mm}(A) < k$, there is a separator of A and $\overline{A}$ that intersect by less than k parts of any witness. In the procedure TRIM_{COL} below we use this to trim the number of certificates to store at each step of the algorithm; if two certificates "projected" to the separator of A and $\overline{A}$ is the same partition, we only store one of them. This results in less than k^k certificates to store.

For two t-partitions P, Q, we say that we *merge* P and Q when we generate a new partition R by pairwise combining the parts (by union) of P with the parts of Q in such a way that R is a t-partition where each part is an independent set. If P and Q can be merged, we say that P and Q are *mergeable*.

Lemma 22. *Given two t-partitions P and Q, deciding whether P and Q are mergeable, and merging P and Q if they are, can be done in polynomial time.*

Proof. We can check whether P and Q are mergeable by reducing it to deciding whether a bipartite graph has a perfect matching: We generate a bipartite graph $B = (P, Q)$ where each vertex/part $p \in P$ is adjacent to $q \in Q$ if and only if $p \cup q$ is an independent set. When P and Q partition sets that do not intersect, then we can merge P and Q by for each edge in the matching, combine the respective parts the edge is incident with. If there is no perfect matching in B, then that must mean there is no way of pairwise combining the parts of P with the parts Q so that all combined parts are independent.

If P contain parts that share elements with parts of Q, then we know these parts must be combined in all merged partitions, so we combine all these sets and then run the above reduction to perfect matching (on the parts that do not have intersecting elements). If a part intersect with more than one other part, then P and Q cannot be merged. $\square$

Lemma 23. *Let A be a subset of $V(G)$, $C \subseteq V(G)$ a separator of A and $\overline{A}$, and P_A and P_C be t-partitions of A and C, respectively. If P_C and P_A merge to a t-partition P'_A, then for any set $B \subseteq (C \cup \overline{A})$ and t-partition P_B of B, P_B is mergeable with P'_A if and only if P_C is mergeable with P_B.*

Proof. For the forward direction, we notice that for any t-partition P' resulting from merging P'_A with P_C, the t-partition $P^* = P' \cap (C \cup B)$ is a result of merging $P'_A \cap (C \cup B) = P_C$ with $P_B \cap (C \cup B) = P_B$, and hence P_C and P_B are mergeable.

We will prove the backwards direction by constructing a partition P' which is the result of merging P'_A with P_B. As P_C partitions exactly the set C, and both P_A merge with P_C to a t-partition P'_A and P_B merge with P_C to a t-partition P'_B, there is an ordering of the parts of P'_A and P'_B so that the i-th part of P'_A intersected with C equals the i-th part of P'_B intersected with C. We let P' be the multiset resulting from pairwise combining the i-th part of P'_A with the i-th part of P'_B. As the only vertices that occur in parts of both P'_A and of P'_B is the set of vertices C, by combining the parts based on the ordering we described, we have made sure that each vertex appear in exactly on part of P', and hence P' is a t-partition.

To show that each part in P' is an independent set, we assume by contradiction that two adjacent vertices x and y are in the same part of P'. By how we constructed P', no two vertices in different parts in either P'_A or P'_B will be in the same part in P'. Therefore, as P'_A partitions $A \cup C$ into parts that are independent sets, and P'_B does the same for $B \cup C$, if x and y are adjacent and in the same part, one of them must be in $A \setminus C$ and the other in $B \setminus C$. However, C disconnects A and B, and hence no edge exists between vertices in $A \setminus C$ and $B \setminus C$, contradicting that x and y are adjacent. $\square$

Procedure $\textsc{Trim}_{\textsc{col}}$ (on input $S \subseteq \text{cert}(A)$, with $A \subseteq V(G)$)

---

remove from S the partitions containing parts that are not independent sets
if $(A, \overline{A})$ is a split **then**
    mark one certificate $P' \in S$ where
        the number of non-empty parts in $N(\overline{A}) \cap P'$ is minimized
else
    $C \leftarrow$ minimum vertex cover of A
    for each t-partition P_C of C **do**
        mark one certificate $P'_C \in S$ which is mergeable with P_C
return the marked certificates of S

Lemma 24. *The procedure* TRIM_{COL} *with input* $S \subseteq A$ *for* $A \subseteq V(G)$ *returns a set* $S' \preceq_A S$ *of size at most* $\text{sm}(A)^{\text{sm}(A)}$, *and has a runtime of* $\mathcal{O}^*(|S|\,\text{sm}(A)^{\text{sm}(A)})$.

Proof. The runtime is correct as a result of the fact that merging (Lemma 22) takes polynomial time to execute, and it produces a set $S' \subseteq S$ of cardinality at most $1 = \text{sm}(A)^{\text{sm}(A)}$ if $(A, \overline{A})$ is a split, and cardinality at most $\text{mm}(A)^{\text{mm}(A)} = \text{sm}(A)^{\text{sm}(A)}$ otherwise. We now show that $S' \preceq_A S$:

For contradiction, let us assume there exists a certificate $P_w \in \text{cert}(\overline{A})$ so that for a certificate $P \in S$ the set $\oplus(P_w, P)$ contains a witness, while $\oplus(\{P_w\}, S')$ does not contain a witness. Let us first assume $(A, \overline{A})$ is a split:

As $(A, \overline{A})$ is a split, each part of P is either adjacent to all of $N(A)$, or not adjacent to $\overline{A}$ at all. Let z be the number of parts in P that are adjacent to all of $N(A)$. This also means z is a lower bound to the number of parts in P_w that do not have neighbours in A. In TRIM_{COL} we mark (and thus output) one certificate P' where at most z parts have neighbours $\overline{A}$. Thus for each of the at most z parts in P' that are adjacent to A, there is a part in P_w not adjacent to any vertex in A. So, we can conclude that P' and P_w can be merged together to form a witness. This contradicts that $\oplus(\{P_w\}, S')$ does not contain a witness when $(A, \overline{A})$ is a split.

Now, let us assume $(A, \overline{A})$ is not a split. Let W be a witness in $\oplus(\{P_w\}, S)$. For the vertex cover C, we have the following smaller t-partitions of W; $P_C = W \cap C$ and $P_W = W \cap C$. By Lemma 23, as C disconnects A from the rest of the graph, any t-partition of A mergeable with P_C is mergeable with P_W. The algorithm assures that for all t-partitions P_C of C, whenever a t-partition in S is mergeable with P_C, at least one t-partition mergeable with P_C exists in S'. From this we can conclude that $\oplus\{P_w\}, S'$ preserves $\oplus(\{P_w\}, S)$. $\square$

Procedure JOIN_{col}(on node w with children a, b and $A_1 = V_a, A_2 = V_b$
and $A = A_1 \cup A_2$ and given $S_1 \preceq_{A_1} cert(A_1)$ and
$S_2 \preceq_{A_2} cert(A_2)$)

$S \leftarrow \emptyset$
$C_1, C_2 \leftarrow$ minimum vertex cover of $G[A_1, \overline{A_1}]$ and $G[A_2, \overline{A_2}]$, respectively

for $P_1 \in S_1$ and $P_2 \in S_2$ **do**
$\quad N_1 \leftarrow$ parts of P_1 not adjacent to $N(A_1)$
$\quad N_2 \leftarrow$ parts of P_2 not adjacent to $N(A_2)$

$\quad$**if** both $(A_1, \overline{A_1})$ and $(A_2, \overline{A_2})$ are splits **then**
$\quad\quad$**for** $0 \leq z \leq \min\{|N_1|, |N_2|\}$ **do**
$\quad\quad\quad$add to S a t-partition generated (if possible) by
$\quad\quad\quad\quad$merging the two partitions P_1 and P_2 together
$\quad\quad\quad\quad$in such a way that exactly z of the parts of N_1
$\quad\quad\quad\quad$is merged with parts of N_2

else if neither $(A_1, \overline{A_1})$ nor $(A_2, \overline{A_2})$ is a split **then**
    for each t-partition P_c of $C_1 \cup C_2$ **do**
        if both P_1 and P_2 are mergeable with P_c **then**
            $P_1^c \leftarrow$ merge P_c and P_1
            $P' \leftarrow$ merge P_1^c and P_2
            add $P' \cap (A \cup B)$ to S

    else
      assign $s, r \in \{1, 2\}$ so that $(A_s, \overline{A_s})$ is the split and $r \neq s$
      for each t-partition P_c of C_r mergeable with P_r **do**
        for each subset Q of the non-empty parts of P_c **do**
            $P_q \leftarrow$ merge (if possible) P_c with P_s so that Q is exactly the
                non-empty parts of P_c that get combined with N_s
            $P' \leftarrow$ the t-partition generated by merging P_q with P_s
            add to S the t-partition $P' \cap (A \cup B)$

return $\text{TRIM}_{col}(S)$

Lemma 25. *Procedure* JOIN_{col} *is correct and runs in time* $\mathcal{O}^*(|S_1| |S_2| k^{3k})$, *producing a set S of cardinality* $\mathcal{O}(k^k)$.

Proof. We will go through the three cases in the algorithm (when zero, one or two out of the two cuts $(A_1, \overline{A_1})$ and $(A_2, \overline{A_2})$ are splits), and show that for each pair (P_1, P_2) of certificates in $S_1 \times S_2$, the output of the algorithm preserves $\oplus(P_1, P_2)$. As the last line of the algorithm assures that the output preserves S (by Lemma 24) and the cardinality of the output is of the correct size, we only need to show that S preserves $\oplus(P_1, P_2)$ and that the runtime is correct.

Suppose neither of the two cuts are splits. In this case, whenever there exists a certificate of $\oplus(P_1, P_2)$ mergeable with a t-partition P_C of the separator $C = C_1 \cup C_2$ of A and $\overline{A}$, the algorithm assures that there is going to be at least one t-partition of A in S mergeable with P_C. By Lemma 23, this means S is going to preserve $\oplus(P_1, P_2)$.

If both of the two cuts are splits, then for each part p_i of every t-partition of A the neighbourhood $N(p_i) \setminus A$, must either be empty, $N(A_1)$, $N(A_2)$, or $N(A)$. In this case the algorithm generates, for each possible z so that there exists an important certificate $P^* \in \oplus(P_1, P_2)$ where the number of parts with empty neighbourhoods is exactly z (and thus, the number of parts with neighbourhood equal $N(A_1)$, $N(A_2)$ and $N(A)$, is $|N_2| - z$, $|N_1| - z$, and $t - |N_1| + |N_2| + z$, respectively), generates a t-partition of independent sets where there the same number of parts with each of the particular four neighbourhoods is equal to the number of parts of each neighbourhood for P^*. We can observe that when two t-partitions over the same set both consist of only independent sets, and there is a correspondence between the parts of both partitions so that the neighbourhood in $\overline{A}$ of each pair of corresponding parts is the same, the two partitions are mergeable to exactly the same t-partitions of $\overline{A}$. Therefore, the

set of t-partitions the algorithm generates in the case when both cuts are splits, preserves $\oplus(P_1, P_2)$.

If exactly one of the cuts are splits, let us assume without loss of generality that $(A_1, \overline{A_1})$ is the split. As above, each part of P_1 is one of two types; adjacent to $N(A_1)$, or not adjacent to $N(A_1)$. When C is a vertex cover of $G[A_2, \overline{A_2}]$, for each important certificate $P^* \in \oplus(P_1, P_2)$, we have a t-partition $P_C = P^* \cap C$ of C. Of the parts in P_c, some subset Z of the non-empty parts get combined with the parts of P_1 that do not have any neighbours in $N(A_1)$ and the rest of the non-empty parts get combined with the other type of parts in P_1. Since C disconnects A_2 from the rest of the graph, there must be a bijection from each part in P^* to parts with the exact same neighbourhood in $N(A)$ for the t-partition generated by the algorithm in the last if/else case, for $Q = Z$. This, in turn, means that the two partitions preserve each other, and so we can conclude that the set of certificates the algorithm produces preserves $\oplus(P_1, P_2)$.

The runtime of the algorithm we get as follows: Excluding polynomials of n, we get the worst case runtime when neither A_1 nor A_2 are splits. Then S grows to be as large as k^{2k} (the number of t-partitions of $C_1 \cup C_2$) for each pair of certificates in $S_1 \times S_2$. This implies a runtime of $\mathcal{O}^*(|S_1||S_2|k^{2k})$ for the entire part before the execution of TRIM_{COL}, which by Lemma 24 takes $\mathcal{O}^*(|S_1||S_2|k^{3k})$-time given the size of S. $\qquad\square$

Theorem 26. *Given a graph G and branch decomposition (T, δ) of sm-width k, we can solve* CHROMATIC NUMBER *in time* $\mathcal{O}^*(k^{5k})$.

Proof. In Lemma 25 we show JOIN_{col} is correct and runs in time $\mathcal{O}^*(|S_1||S_2|k^{3k})$, producing a set S of cardinality $\mathcal{O}(k^k)$. So, using RECURSIVE with JOIN_{col}, we know the size of both of the inputs of JOIN_{col} is at most the size of its output, i.e., $|S_1|, |S_2| \leq \mathcal{O}^*(k^k)$. So, each call to RECURSIVE has runtime at most $\mathcal{O}^*(k^{5k})$. As there are linearly many calls to RECURSIVE and there is a polynomial time verifier for the certificates RECURSIVE produces, by the definition of $\preceq$, the total runtime is also bounded by $\mathcal{O}^*(k^{5k})$. To solve CHROMATIC NUMBER when we have an algorithm for t-COLORING, we simply run the t-COLORING algorithm for each value of $t \leq n$, giving the same runtime for CHROMATIC NUMBER when excluding polynomial factors of n. $\qquad\square$

4.4   Edge Dominating Set

4.4.1   The problem.

The decision problem t-EDGE DOMINATING SET asks for an input graph G, whether there exists a set $E' \subseteq E(G)$ of cardinality at most t, so that for each edge $e \in E(G)$ either $e \in E'$ or e shares an endpoint with an edge $e' \in E'$. We say that an edge $e' \in E'$ *dominates* $e \in E(G)$ if e and e' share an endpoint.

4.4.2   The certificates and $\oplus$.

For t-EDGE DOMINATING SET, we define $\text{cert}(X)$ for $X \subseteq V(G)$ to be all subgraphs of $G[X]$. This might seem odd, as we are looking for a set of edges.

However, for a certificate $G' \in \text{cert}(X)$, the set $V(G')$ is going to make the algorithm simpler. A witness will be a subgraph G_w such that $E(G_w)$ is an edge dominating set of size at most t and each vertex of $V(G_w)$ is incident with an edge of $E(G_w)$. Checking the latter can obviously be done in polynomial time and checking that $E(G_w)$ is an edge dominating set of size at most t can also be done in polynomial time since t-EDGE DOMINATING SET is in NP.

For disjoint sets $A, B \subseteq V(G)$ and certificates $G_A \in \text{cert}(A)$ and $G_B \in \text{cert}(B)$, we define $\oplus(G_A, G_B)$ to be the set

$$\{G_A + G_B + E' : E' \subseteq E(G[V(G_A), V(G_B)])\}.$$

We can see that this definition satisfies $\text{cert}(A \cup B) = \oplus(\text{cert}(A), \text{cert}(B))$.

4.4.3 The Join$_{\text{eds}}$ function.

Given a graph G' and vertex $v \in V(G')$, we say that v is an *isolated* vertex of G' if it is not incident with any edge of $E(G')$. If a vertex is not isolated, it is *non-isolated*. We say a set of edges E' *span* a set of vertices X if $X \subseteq V(E')$. We denote by $I(G')$ the isolated vertices of G'.

We say that a certificate $G' \in \text{cert}(A)$ is *locally correct* if all edges in $G[A]$ have an endpoint in $V(G')$ and all the isolated vertices of $V(G')$ are in $N(\overline{A})$. A certificate is *locally incorrect* if it is not locally correct. We see that a certificate in $\text{cert}(A)$ which is locally incorrect cannot also be an important certificate as there exists an edge in $G[A]$ which is not dominated by edges in $E(G')$ and cannot be dominated by edges in $G[A, \overline{A}]$.

We observe that for two locally correct certificates $G_1, G_2 \in \text{cert}(A)$, if $N(\overline{A}) \cap G_1 = N(\overline{A}) \cap G_2$ and the isolated vertices of G_1 equal those of G_2, then $G_1 \preceq_A G_2$ if $|E(G_1)| \leq |E(G_2)|$.

Lemma 27. *Given a graph G without isolated vertices and a subset $A \subseteq V(G)$, a minimum cardinality set $X \subseteq E(G)$ of edges spanning the vertices A can be found in polynomial time.*

Proof. Let M be a maximum matching of $G[A]$. Any set spanning A must be of size at least $|M| + (|A| - 2|M|) = |A| - |M|$, as otherwise there must exist a matching in $G[A]$ larger than M. Let R be the set of vertices in A not incident with any edge in M. As no vertex of G is isolated, each vertex in R is incident with at least one edge in G. Thus, we can easily find a set E_R of $|R|$ edges spanning R. We claim that the set $X = M \cup E_R$ is a minimum cardinality set of edges spanning A; Clearly X spans A, as each vertex not spanned by M is by definition spanned by E_R. Furthermore, the size of R is exactly $|A| - 2|M|$, so the size of X is $|M| + (|A| - 2|M|) = |A| - |M|$. Hence, the set X is a minimum cardinality set spanning A. $\square$

As usual, our join-procedure will consist of a part where we join together pairs of certificates, imitating $\oplus$, and a part where the size of the output is reduced using a trim-procedure that ensures the output is preserving. For the EDGE DOMINATING SET-problem, the trimming part consists of two procedures;

one for when the cut in question is a split, and one for when it is not a split. We first present the simplest of the two, namely the one used when the cut is a split (TRIM_{EDS}-SPLIT).

Procedure TRIM_{EDS}-SPLIT (on $S \subseteq \text{cert}(A)$ where $(A, \overline{A})$ is a split)

---

remove from S all locally incorrect certificates
for each $0 \leq x_1, x_2 \leq n$ **do**
    mark one certificate $G' \in S$ of minimized $|E(G')|$ where
        x_1 equals $|I(G')|$, and
        x_2 equals $\left| V(G') \cap N(\overline{A}) \right|$
return all the marked certificates in S

Lemma 28. *The procedure* TRIM_{EDS}-SPLIT *on* $S \subseteq \text{cert}(A)$ *for* $A \subseteq V(G)$ *where* $(A, \overline{A})$ *is a split, returns a set* $S' \subseteq S$ *so that* $S' \preceq_A S$ *and* $|S'| \in \mathcal{O}(n^2)$ *in* $\mathcal{O}^*(|S|)$-*time.*

Proof. The algorithm clearly does $(n+1)^2$ number of iterations in the for-loop, and for each of these iterations it iterates through the list S. For each element in S, it does a polynomial amount of work, so the total runtime is $\mathcal{O}^*(|S|)$. Furthermore, at most one element is marked to be put in the output at each iteration, so the output of the algorithm is of size $\mathcal{O}(n^2)$.

For the correctness of the algorithm, notice that for any two locally correct certificates $G_1, G_2 \in \text{cert}(A)$, if $|I(G_1)| = |I(G_2)|$ and $|V(G_1) \cap N(\overline{A})| = |V(G_2) \cap N(\overline{A})|$, then G_1s preserves G_2. This is because each vertex in $N(\overline{A})$ is adjacent to exactly the same vertices of $\overline{A}$, and so for any set $X_2 \subseteq N(\overline{A})$, if there is a set of edges spanning $X \subset \overline{A}$ and $X_1 \subseteq N(\overline{A})$, there is also a set of edges of the same cardinality spanning X_2 and X as long as $|X_2| = |X_1|$. $\square$

Now we describe the trim-procedure for when the respective cut is not a split (TRIM_{EDS}-NON-SPLIT). This procedure is more complicated than TRIM_{EDS}-SPLIT. The core idea of the procedure is that when $G_a + G_{\overline{a}} + E_w \in \oplus(G_a, G_{\overline{a}})$ is a witness, then also for any locally correct certificate G_i, the certificate $G_i + G_{\overline{a}} + E'_w \in \oplus(G_i, G_{\overline{a}})$ is also a witness, as long as $V(G_i) \cup V(G_{\overline{a}})$ is a vertex cover of $G[A, \overline{A}]$, E'_w spans $I(G_i)$ and the vertices in $V(E_w) \cap \overline{A}$, and $|E(G_i)| + |E'_w| \leq |E(G_a)| + |E_w|$.

Procedure TRIM_{EDS}-NON-SPLIT (on $S \subseteq \text{cert}(A)$)

---

remove from S all locally incorrect certificates
$C \leftarrow$ minimum vertex cover of $G[A, \overline{A}]$
for all $Q \subseteq R \subseteq C$ **do**
    $R_a \leftarrow R \cap A$
    $R_{\overline{a}} \leftarrow R \cap \overline{A}$
    mark one certificate $G_i \in S$ minimizing $|E_i| + |E(G_i)|$ where:
        $V(G_i) \cap C = R_a$,
        $C \setminus (R_{\overline{a}} \cup A) \subseteq N(V(G_i))$, and
        $E_i \leftarrow$ a minimum subset of $E(G[V(G_i), R_{\overline{a}}])$ spanning $(I(G_i) \cup R_{\overline{a}}) \setminus Q$
    (if no such G_i exists, then don't mark any certificate)
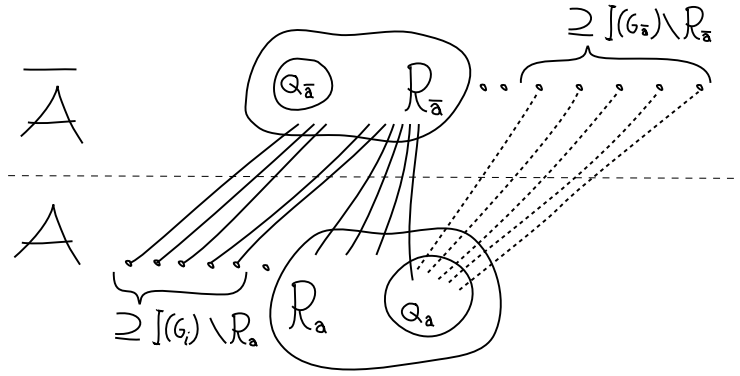return the set of all the marked certificates in S

Figure 5: As described in the proof of Lemma 29. The dotted lines constitute the set E_I.

Lemma 29. *Procedure* TRIM_{EDS}-NON-SPLIT *produces a set $S' \preceq_A S$ of size at most $3^{\text{mm}(A)}$ in time $\mathcal{O}^*(3^{\text{mm}(A)}|S|)$.*

Proof. The size of S' is apparent from the fact that we iterate through all sets $Q \subseteq R \subseteq C$ (at most $3^{\text{mm}(A)}$ triples), and mark one certificate in S to later be put in the output set. When deciding which certificate to mark, we possibly search through the entire set S, and do a polynomial amount of work on each certificate in S to check if it is the certificate that should be marked in this iteration. This gives a total runtime of $\mathcal{O}^*(3^{\text{mm}(A)}|S|)$.

By the definition of the $\preceq_A$-relation, we have $S' \preceq_A S$ if for every important certificate $G_a \in S$ and $G_{\overline{a}} \in \text{cert}(\overline{A})$ so that $\oplus(G_a, G_{\overline{a}})$ contains a witness, the set $\oplus(S', \{G_{\overline{a}}\})$ also contains a witness.

Let $G_w = G_a + G_{\overline{a}} + E_w$ be a witness of $\oplus(G_a, G_{\overline{a}})$. We show that the algorithm will mark a certificate G_i so that the set $\oplus(G_i, G_{\overline{a}})$ also contains a

witness, thereby establishing the lemma. By our definition of $\oplus$, the set E_w is a subset of edges from $E(G[V(G_a), V(G_{\overline{a}})])$. Let R' be the vertices of $V(G_w) \cap C$ and let E_I be the edges of E_w where one endpoint is in $C \cap A$ and the other endpoint is in $\overline{A} \setminus C$. Note that E_I must span the set $I(G_{\overline{a}}) \setminus R'$. We let Q_a be the set of vertices in $R' \cap A$ that are incident with edges of E_I, and let $Q_{\overline{a}}$ be the vertices of $R' \cap \overline{A}$ not incident with any edge in E_w (see Figure 5).

At the iteration of the algorithm where $Q = Q_a \cup Q_{\overline{a}}$ and $R = R'$, the algorithm marks a certificate G_i to be put into S'. This means for the certificate G_i there is a set $E_i \subseteq E(G[V(G_i), R_{\overline{a}}])$ spanning $I(G_i) \setminus Q_a$ and $R_{\overline{a}} \setminus Q_{\overline{a}}$. Furthermore, we have $|E_i| + |E(G_i)| \leq |E_w \setminus E_I| + |E(G_a)|$, as otherwise the algorithm would prefer marking G_a over marking G_i.

We will now show that the certificate $G'_w = G_i + G_{\overline{a}} + E_I + E_i \in \oplus(G_i, G_{\overline{a}})$ is a witness. To do this, it will be enough to show the following three things: (1) $V(G'_w)$ is a vertex cover of the original graph G, (2) $|E(G'_w)| \leq |E(G_w)|$, and (3) $I(G'_w)$ is empty.

The first point, that $V(G'_w)$ is a vertex cover, we get from the fact that G'_w is a vertex cover of $G[A, \overline{A}]$ and that as both G_i and $G_{\overline{a}}$ are locally correct $V(G_i)$ and $V(G_{\overline{a}})$ are vertex covers of $G[A]$ and $G[\overline{A}]$, respectively. So, G'_w is a vertex cover of $G[A] + G[\overline{A}] + G[A, \overline{A}] = G$.

The second point, $|E(G'_w)| \leq |E(G_w)|$, holds by the following inequality.

$$
\begin{aligned}
|E(G'_w)| &= |E(G_i)| + |E_i| + |E(G_{\overline{a}})| + |E_I| \\
&\leq |E(G_a)| + |E_w \setminus E_I| + |E(G_{\overline{a}})| + |E_I| = |E(G_w)| \ .
\end{aligned}
$$

What remains to prove is the third point, that $I(G'_w)$ is empty. To do this, we will show that each vertex in $I(G_i)$ and $I(G_{\overline{a}})$ is spanned by the edges $E_I + E_i$. We defined the vertices of $Q_{\overline{a}}$ to not be incident with any edges of E_w, and we know $I(G_w)$ is empty, so we can conclude that $Q \cap I(G_{\overline{a}})$ is empty also. This means that as E_i spans the set $R \cap \overline{A} \setminus Q$, in fact E_i spans all the vertices of $I(G_{\overline{a}}) \cap R$. We defined E_I to be all the edges of E_w with endpoints in $\overline{A} \setminus R$, and E_w spans $I(G_{\overline{a}})$, so E_I must span $I(G_{\overline{a}}) \setminus R$. From this we conclude that $I(G'_w) \setminus A$ is empty. Now we must show that also $I(G'_w) \cap A$ is empty. We know E_i spans all the vertices of $I(G'_w) \cap A \setminus Q_a$ by definition, and the set Q_a is exactly the set of vertices in A that are incident with edges of E_I. So, $E_I \cup E_i$ also spans $I(G'_w) \cap A$, and hence $I(G'_w) = \emptyset$. $\qquad\square$

Next, we give the following lemma, which will be used directly in the join operation of our algorithm solving t-Edge Dominating Set. It will be helpful in reducing the number of certificates needed to ensure we have a set preserving $\oplus(G_1, G_2)$ for given certificates G_1 and G_2. The idea behind this lemma is that for any witness $G_1 + G_2 + G_3 + E_w$, we can substitute the set E_w with another set E'_w spanning all the isolated vertices of G_1, G_2 and G_3 to produce a new witness, as long as $|E'_w| \leq |E_w|$. This means that a lot of different certificates $G_1 + G_2 + E_t \in \oplus(G_1, G_2)$ will generate a witness when combined with the same certificate G_3 as long as there is a set E'_t so that $E'_t + E_t$ spans the isolated vertices of G_1, G_2 and G_3. Having this in mind, we are able to limit the number of certificates needed to preserve $\oplus(G_1, G_2)$ greatly.

Lemma 30. *For any disjoint sets $A_1, A_2 \subseteq V(G)$ and certificates $G_1 \in \mathrm{cert}(A_1)$ and $G_2 \in \mathrm{cert}(A_2)$, we can in $\mathcal{O}^*(2^{\mathrm{sm}(A_1)} + 2^{\mathrm{sm}(A_2)})$-time compute two set families $\mathcal{F}(G_1, G_2) \subseteq 2^{V(G_1)}$ and $\mathcal{F}(G_2, G_1) \subseteq 2^{V(G_2)}$ where the following holds:*

1. *$|\mathcal{F}(G_1, G_2)| \leq \max \left\{ 2^{\mathrm{sm}(A_1)}, n \right\}$ and $|\mathcal{F}(G_2, G_1)| \leq \max \left\{ 2^{\mathrm{sm}(A_2)}, n \right\}$, and*

2. *there is a set $S \subseteq \oplus(G_1, G_2)$ preserving $\oplus(G_1, G_2)$, where for each certificate $G_1 + G_2 + E_s \in S$ the set E_s has $V(E_s) \cap A_1 \in \mathcal{F}(G_1, G_2)$ and $V(E_s) \cap A_2 \in \mathcal{F}(G_2, G_1)$.*

Proof. Let $A_3 = \overline{A_1 \cup A_2}$. We will give a construction of the set families $\mathcal{F}(G_1, G_2)$ and show that for any certificate $G_z = G_1 + G_2 + E_z$ in $\oplus(G_1, G_2)$, there is a certificate $G'_z = G_1 + G_2 + E'_z$ in $\oplus(G_1, G_2)$ so that $G'_z \preceq_{\overline{A_3}} G_z$ and that $V(E'_z) \cap A_2 = V(E_z) \cap A_2$ and $V(E'_z) \cap A_1 \in \mathcal{F}(G_1, G_2)$. By similar construction and argument for $\mathcal{F}(G_2, G_2)$, we can conclude that constraint (*2.*) holds for the two constructed set families. That the two set families can be constructed within the proposed time bound and that the size of the sets are as stated in constraint (*1.*) is evident from how we are going to construct them.

Suppose for certificate $G_z = G_1 + G_2 + E_z \in \oplus(G_1, G_2)$ there is some certificate $G_3 \in \mathrm{cert}(A_3)$ so that we have a witness $G_w = G_1 + G_2 + G_3 + E_w$ in $\oplus(G_3, G_z)$. Let C be a vertex cover of $G[A_1, \overline{A_1}]$. We are particularly interested in the following three parts of E_w incident with A_1 (see Figure 6).

- E_1: the subset of edges that go from $A_1 \cap C$ to A_2,

- E_2: the subset of edges between $A_1 \setminus C$ and $A_2 \cap C$, and

- E_3: the edges that go from $A_1 \setminus C$ to $A_3 \cap C$.

We denote by R_1, R_2 and R_3 the endpoints $A_1 \cap V(E_1)$, $A_2 \cap V(E_2)$, and $A_3 \cap V(E_3)$, respectively.

As G_w is a witness, the edges in E_w must span all the isolated vertices of G_1, G_2, and G_3. We notice that the edges $E(G_w) \setminus (E_2 \cup E_3)$ span all the vertices of $V(G_w)$ except possibly some vertices in $I(G_1) \cup R_2 \cup R_3$. Therefore, for any subset $E' \subseteq E(I(G_1) \setminus R_1, R_2 \cup R_3)$ spanning $(I(G_1) \setminus R_1) \cup R_2 \cup R_3$, it will be the case that $E(G_w) + E' - E_2 - E_3$ spans $V(G_w)$. Furthermore, if $|E'| \leq |E_2 \cup E_3|$, then the certificate $G'_z = G_w - E_2 - E_3 + E'$ will be a witness. To ensure that constraint (*2.*) is satisfied, it thus suffice to ensure that the vertices in A_1 adjacent to vertices in A_2 in G'_z constitute a set in $\mathcal{F}(G_1, G_2)$. For G'_z, these vertices are exactly $R_1 \cup (V(E') \cap A_1)$. What we notice is that E' only depended on G_1, R_1, R_2 and R_3. So, if for each choice of R_1, R_2, and R_3, we compute the E' as we did above, we will have made a set family satisfying constraint (*2.*) and which can be computed within a polynomial factor in n of its size.

What we may now observe, is that since $R_1 \subseteq C \cap A_1$, $R_2 \subseteq C \cap A_2$ and $R_3 \subseteq C \cap A_3$, the different possibilities for R_1, R_2 and R_3 combined is at most $2^{|C|}$. This means the size of the set family is at most $2^{\mathrm{mm}(A_1)}$ and the time to compute it is $\mathcal{O}^*(2^{\mathrm{mm}(A_1)})$.

If $(A_1, \overline{A_1})$ is a split, however, it might be the case that $\mathrm{sm}(A_1) < \mathrm{mm}(A_1)$. But when it is a split, as the neighbourhood of each vertex in $I(G_1)$ is the same, as long as there is a set $S_i \in \mathcal{F}(G_1, G_2)$ of i vertices maximizing $\max\{|S_i \cap I(G_1)|\}$ for each $i \in \{0, \ldots, |V(G_1)|\}$, the set family $\mathcal{F}(G_1, G_2)$ satisfies constraint (2.). The size constraint follows from the fact that we only need at most n sets S_i, and clearly we can compute the set family in the runtime stated, as it only takes polynomial amount of time to generate S_i greedily for each i. $\qquad\square$
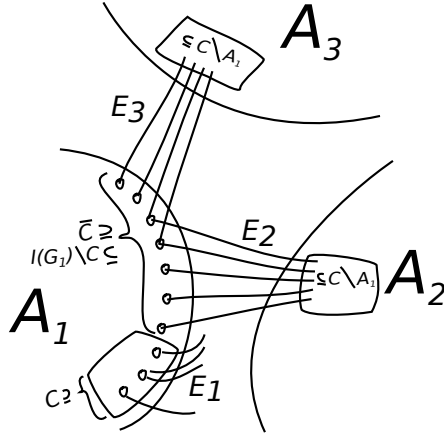

Figure 6: As described in proof of Lemma 30. The vertices in each of the three rectangles are subsets of the vertex cover C of A_1, so these sets can be of at most $2^{|C|}$ possibilities.

Procedure $\mathrm{JOIN}_{\mathrm{EDS}}$ (on node w with children a, b and $A_1 = V_a, A_2 = V_b$
                   and $A = A_1 \cup A_2$ and given $S_1 \preceq_{A_1} \mathrm{cert}(A_1)$ and
                   $S_2 \preceq_{A_2} \mathrm{cert}(A_2)$)

$S \leftarrow \emptyset$
for each $G_1 \in S_1$, $G_2 \in S_2$ **do**
$\quad \mathcal{V}_1 \leftarrow \mathcal{F}(G_1, G_2)$ from Lemma 30
$\quad \mathcal{V}_2 \leftarrow \mathcal{F}(G_2, G_1)$ from Lemma 30
$\quad$ **for** each $X_1 \in \mathcal{V}_1$, $X_2 \in \mathcal{V}_2$ **do**
$\quad\quad E' \leftarrow$ minimum sized subset of $E(G[X_1, X_2])$ spanning $X_1 \cup X_2$
$\quad\quad$ add to S the certificate $G_1 + G_2 + E'$

if $(A, \overline{A})$ is a split **then return** $\mathrm{TRIM}_{\mathrm{EDS}}\text{-}\mathrm{SPLIT}(S \subseteq \mathrm{cert}(A))$
else return $\mathrm{TRIM}_{\mathrm{EDS}}\text{-}\mathrm{NON}\text{-}\mathrm{SPLIT}(S \subseteq \mathrm{cert}(A))$

Lemma 31. *The algorithm* JOIN_{EDS} *is correct and runs in time* $\mathcal{O}^*(|S_1|\,|S_2|\,12^k)$, *producing a set S of cardinality* $\mathcal{O}(n^2 + 3^k)$ *when both* $\text{sm}(A_1)$, $\text{sm}(A_2)$, *and* $\text{sm}(A_1 \cup A_2)$ *is at most k.*

Proof. For $G_1 \in \text{cert}(A_1)$ and $G_2 \in \text{cert}(A_2)$, if $G^* = G_1 + G_2 + E^* \in \oplus(G_1, G_2)$ and $G' = G_1 + G_2 + E'$ where $V(E') = V(E^*)$, then $G' \preceq_{A_1 \cup A_2} G^*$ as long as $|V(E')| \le |V(E^*)|$. Therefore, the set of certificates S generated in the first part of the algorithm must preserve the set "S" from point 2. in the statement of Lemma 30, which in turn implies that $S \preceq_{A_1 \cup A_2} \oplus(S_1, S_2)$. By Lemma 28 and Lemma 29 this means the output of the algorithm is a set of size at most $n^2 + 3^{\text{sm}(A_1 \cup A_2)}$ that preserves $\oplus(S_1, S_2)$.

From Lemma 30, for each pair of certificates, it takes $\mathcal{O}^*(2^k)$ time to compute $\mathcal{F}(G_1, G_2)$ and $\mathcal{F}(G_2, G_1)$, and for each pair we generate at most $|\mathcal{F}(G_1, G_2)|\,|\mathcal{F}(G_2, G_1)|$ certificates. So, before the call to one of the TRIM_{EDS}-procedures, the algorithm uses $\mathcal{O}^*((2^{2k})|S_1|\,|S_2|)$-time and S contains at most $(2^{2k})|S_1|\,|S_2|$ certificates. The total runtime, including the call to the respective TRIM_{EDS}-procedure, must then by Lemma 28 and Lemma 29 be $\mathcal{O}^*(3^k 2^{2k}\,|S_1|\,|S_2|)$. □

Theorem 32. *Given a graph G and branch decomposition (T, δ) of sm-width k, we can solve* EDGE DOMINATING SET *in time* $\mathcal{O}^*(3^{5k})$.

Proof. In Lemma 31 we showed that the procedure JOIN_{EDS} is correct and runs in time $\mathcal{O}^*(|S_1|\,|S_2|\,12^k)$, producing a set S of cardinality $\mathcal{O}(n^2 + 3^k)$. So, using RECURSIVE with JOIN_{EDS}, we know the size of both of the inputs of JOIN_{EDS} is at most the size of its output, i.e., $|S_1|, |S_2| \le \mathcal{O}^*(3^{5k})$. So, each call to RECURSIVE has runtime at most $\mathcal{O}^*(3^k)$. As there are linearly many calls to RECURSIVE and there is a polynomial time verifier for the certificates RECURSIVE produces, by the definition of $\preceq$, the total runtime is also bounded by $\mathcal{O}^*(3^{5k})$. To solve EDGE DOMINATING SET, we run the t-EDGE DOMINATING SET algorithm for all values of $t \le n$ and hence this is also solvable in $\mathcal{O}^*(3^{5k})$ time as we exclude polynomials of n. □

5   Graphs of bounded sm-width

In this section we discuss graph classes of bounded sm-width.

Proposition 33. *If treewidth is bounded then sm-width is bounded* $(\text{smw}(G) \le tw(G) + 1)$ *which in turn means that clique-width is bounded.*

Proposition 34. *If twin-cover is bounded then clique-width and sm-width is bounded* $(\text{smw}(G) \le tc(G))$.

Proposition 35. *Cographs, the graphs of clique-width at most two, have sm-width one, while distance-hereditary graphs have clique-width at most three and sm-width one.*

Proof of Proposition 33,34 and 35. Firstly, by the definition of sm-width and MM-width it is clear that for any graph G we have $\text{smw}(G) \leq \text{mmw}(G)$ and it then follows by Theorem 1 (by Vatshelle [18]) that $\text{smw}(G) \leq \text{tw}(G) + 1$. Let us argue that if sm-width is bounded on a class of graphs, then so is clique-width. Rao [16] shows that the clique-width of a graph is at most twice the maximum clique-width of all prime graphs in a split decomposition of the graph. By Observation 11 and the fact that every prime graph is an induced subgraph we know that sm-width of a graph is at least as large as the maximum sm-width of all prime graphs. Theorem 1 and the fact that sm-width of a prime graph is equal to the MM-width of the prime graph, tells us that the sm-width of a prime graph is bounded whenever its treewidth is bounded. Thus, since clique-width is stronger than treewidth it is also stronger than sm-width.

Secondly, twin-cover $tc(G)$ is a graph parameter introduced by Ganian [10] as a generalization of vertex cover that is bounded also for some dense classes of graphs. Gajarský et al [9] in their study of the modular-width parameter $mw(G)$ showed that $mw(G) \leq 2^{tc(G)} + tc(G)$ which in turn implies that if twin-cover is bounded for a class of graphs then also clique-width is bounded. We show that $\text{smw}(G) \leq tc(G)$. Using the definition of $tc(G)$ from [10] it follows that G has a set $S \subseteq V(G)$ of at most $tc(G)$ vertices such that every component C of $G \setminus S$ induces a clique and every vertex in C has the same neighborhood in S. Let $C_1, ..., C_q$ be the components of $G \setminus S$. Take any branch decomposition of G having for each component C_i a subtree T_i such that the leaves of T_i are mapped to the vertices of C_i and also having a subtree T_S whose leaves are mapped to S. The cuts of G induced by an edge of this branch decomposition are of three types depending on where the edge is: if it is inside the tree of S the cut has a maximum matching of size at most $|S| \leq tc(G)$; if it is inside a tree T_i the cut is a split; otherwise the cut has a maximum matching of size at most $|S| \leq tc(G)$.

Thirdly, any cograph is also a distance-hereditary graph and any distance-hereditary graph G has clique-width at most three, see e.g. [12]. Also, G has a branch decomposition of cut-rank one, as shown by Oum [14], which means that all cuts induced by edges of this branch decomposition are splits and hence any distance-hereditary graph has sm-width one. $\square$

There are several classes of graphs of bounded sm-width where no previous results implied FPT algorithms for the considered problems. We now show a class of such graphs, constructed by combining a graph of clique-width at most 3, with a graph of treewidth k and thus clique-width at most $2^{k/2}$, as follows. Let G_1 a distance-hereditary graph and let G_2 be a graph of treewidth k. Let $X \subseteq V(G_1)$ with $|X| \leq k+1$ and $(X, \overline{X})$ a split of G_1, and let $Y \subseteq V(G_2)$ be a bag of a tree decomposition of G_2 of treewidth k. Add an arbitrary set of edges on the vertex set $X \cup Y$. The resulting graph will have sm-width at most $k+1$, a result that basically follows by taking branch decompositions of G_1 and G_2 where X and Y each are mapped as the set of leaves of a subtree, subdividing each of the two edges above these subtrees and adding an edge on the subdivided vertices to make a single branch decomposition of the combined graph.

Note that we can also construct new tractable classes of graphs by combining

several graphs in a tree structure.

6 Conclusions

We have shown that four basic problems, that cannot be FPT parameterized by clique-width unless FPT = W[1], are FPT when parameterized by the split-matching-width of the graph, a parameter whose modelling power is weaker than clique-width but stronger than treewidth. This was accomplished using the theory of split decompositions and the recently introduced MM-width, combined with slightly non-standard dynamic programming algorithms on the resulting decompositions. Graph classes of bounded sm-width will consist of graphs having low treewidth in local parts, with these parts connected together in a dense manner. We have not found references to such graph classes in the litterature. Nevertheless, the appealing algorithmic properties of these graph classes should merit an interest in their further study.

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